Essays on Macroeconomics and Finance

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Dedication

To my parents, Feiying and Guoguang

Abstract

This dissertation consists of two parts. The unifying scheme is to advance our understanding of the nature of financial markets, in particular their macroeconomic implications. In the first essay, I focus on the microstructure of the financial market. In particular, I study how investors' information choices interact over time, and how does this dynamic aspect change the nature of information acquisition in financial markets? In an infinite-horizon framework in which a stock's dividend has a persistent component (stock fundamental), and overlapping generations of investors choose whether to acquire costly information about this time-varying component, I illustrate that information is like bubble in that its value is forward-looking: current investors have more incentives to acquire information if more investors get informed in the future, as the future resale stock price becomes more sensitive to the fundamental. This dynamic complementarity in information acquisition leads to multiple stationary rational-expectation equilibria, despite presence of the classic static substitutability force as in [1]. The dynamic complementarity in information acquisition can be most prominent with intermediate persistence of stock fundamental, or when the public signal is imprecise.

The second essay develops and quantifies a financial theory to account for the macroeconomic phenomenon that the recovery from the Great Recession was much slower than recoveries from other postwar recessions . I propose a standard neoclassical model enriched with a land sector. Land can be used either as a consumption good for the household or as collateral for the firm to finance working capital. The model exhibits two locally stable steady states: one with high capital and high land price, and the other with low capital and low land price. The multiplicity of steady states allows for asymmetric responses to small and large shocks. Large adverse shocks have a much more persistent impact, as they trigger transitions from one steady state to the other. A calibrated version of the model displays significantly delayed recovery upon large adverse shocks and is consistent with various features of the Great Recession and its aftermath.

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Dynamic Complementarity In Information Acquisition

1.1 Introduction

A common explanation for why investors trade in stock markets is that they have access to different information. To motivate heterogeneity in information, it is typically assumed that information is costly to acquire, so that some investors acquire information whereas others do not. This raises interesting questions about information transmission and trading. In particular, how do investors' information acquisition decisions interact with each other? Is information a complement or a substitute? [1], in a static model, provide a classical view to this question: the fact that privately acquired information is partially revealed through prices means that the larger the share of informed investors today, the smaller the return to information acquisition. Thus, information is a *static substitute* in that its value decreases with the share of informed investors *today*. This substitutability in information acquisition leads to a unique equilibrium in [1]. Following [1], the literature predominantly assumes that agents are allowed to acquire information only at the beginning of the economy. What if this assumption is relaxed and dynamic information acquisition is allowed? In particular, how do investors' information choices interact *over time*? How does this dynamic aspect change the nature of information acquisition in financial markets?

This paper fills the gap by endogenizing information choice in an otherwise standard infinite-horizon asymmetric information trading model.[1] There is a single long-lived stock that pays a dividend each period. The dividend is stochastic and consists of a persistent component (the stock fundamental) and a noisy component. The stock's supply also follows some persistent process. There are overlapping generations of investors. Members of each generation, upon their birth, freely observe the entire history of stock prices and dividends. They are then offered an opportunity to become informed, i.e. to observe the history of stock fundamental at some cost.

What determines investors' incentive to acquire information in this dynamic environment? First, the classic static substitutability in information acquisition still presents as investors can freely observe current and past stock prices that partially reveal information. Second, there is a dynamic complementarity force emerging in this infinite-horizon overlapping-generation framework, making the value of information forward looking. That is, the value of information today is increasing in the share of informed investors *in the future*. This is because as more investors get informed in the future, future resale stock prices become more sensitive to the fundamental. This increases the conditional variance of future stock payoff and thus creates more uncertainty for today's uninformed agents (because they cannot perfectly observe today's fundamental). This raises the value of becoming informed today. Like a bubble, this forward-looking self-fulfilling prophecy opens the door to multiplicity in information acquisition in an infinite-horizon framework.

My first main result is that this dynamic complementarity may dominate static substitutability and lead to multiplicity in information acquisition. The intuition is as follows. In this dynamic environment, multiple stationary equilibrium arise if the conditional variance of the stock payoff, and thus the value of information, is increasing in the steady-state share of the informed investors. Now consider a marginal increase in the steady-state share of the informed. This has two opposing effects on the value of information. First, there are more informed investors today. Thus, the current stock price becomes a more precise signal about the fundamental. This tends to reduce the conditional variance of stock payoffs faced by the uninformed and thus the value of information. This is the classical static substitutability effect. The magnitude of this

[1] The physical and information structure is very close to [2], [3], and more recently, [4], except that here the information choice is endogenous.

effect is proportional to the loading of the stock *price* on the fundamental.[2] Second, there are more informed investors in the future. This makes the future *resale* stock price more sensitive to the fundamental and thus increases the conditional variance of stock payoffs. This is the dynamic complementarity effect. The magnitude of this effect is roughly proportional to the loading of the future stock *payoff* on the fundamental.[3]

Since the stock payoff consists of both the *interim dividend payout* and the future resale stock price and thus is *more* sensitive to the fundamental than just the stock price, the dynamic complementarity effect dominates the static substitutability effect, implying an upward-sloping value of information with respect to the steady-state share of informed investors, leading to multiplicity.

It is well known that infinite horizon, overlapping generation models of this type have multiple financial market equilibria when information acquisition is exogenous [5, 6, 3, 7]. To focus on my point, in the main text of the paper I select high-volatility equilibria and within the equilibria I prove that multiplicity in information acquisition exists. The multiplcity result persists to low-volatility equilibria.

My second main contribution is to explore conditions under which the dynamic complementarity, and hence multiplicity in information acquisition prevails. The flexible framework proposed here allows me to examine a wide range of parameters including the persistence of stock fundamental, the persistence of stock supply, and the precision of public signal. It also allows me to examine how the nature of financial market equilibrium shapes the dynamic complementarity in information acquisition. To this end, I confirm some existing findings in the literature [8],[4] as well as offering new insights.

Surprisingly, the paper find that there exists a nontrivial relationship between the persistence of stock fundamental and the strength of dynamic complementarity. This relationship also depends on which *financial market* equilibrium one selects: at the high-volatility equilibrium, higher fundamental persistence always strengthens the dynamic complementarity; whereas at the low-volatility equilibrium the inverse might be true,

[2] Roughly, this is because the variance of the price signal is proportional to the square of the loading coefficients on the fundamental of the current stock price.

[3] Similar to footnote 2, this is because the variance of the stock payoff is proportional to the square of the loading coefficients on the fundamental of the stock payoffs.

[4] For instance, similar to [8], here the complementarity in information acquisition are more prominent when the stock supply is less persistent.

and there may exist a U-shaped relationship where the dynamic complementarity is most prominent with intermediate level of stock fundamental persistence. When stock fundamental becomes more persistent, there are two opposing effects. First, current stock fundamental becomes more predictive of future stock fundamental. This strengthens the dynamic complementarity. Second, there is a *dynamic market learning effect* whereby increasing the fraction of *future* informed investors makes *future* uninformed investors trade more actively on their noisy signals, because they observe a more precise, albietly still noisy, price signal. This introduces more uncertainty faced by the *current* informed and hence reduce the current value of information. The *dynamic market learning effect* weakens the dynamic complementarity in information acquisition. Note that this effect diminishes at the high-volatility equilibrium where the conditional variance of future stock payoff is already extremely high. Thus, adding more uncertainty does not change much the overall level of conditional variance.

The paper also finds that multiplicity is less likely to arise when the precision of the public signal improves. Thus, for a regulator aiming to stabilize asset markets, disclosing more precise public information is helpful as it helps eliminate equilibrium multiplicity in information acquisition. This result contributes to the recent debate on the desirability of the regulatory effort to provide more precise public information, such as the Sarbanes-Oxley Act and, more recently, the Dodd-Frank Act.

The paper is structured as follows. Section 1.2 sets up the model economy. Section 1.3 describes the formal steps of the proof. Section 1.4 provides intuition for the main step of the proof. Section 1.5 conducts comparative statics excercises. Section 1.6 concludes.

Literature Review The dynamic complementarity in information acquisition is reminiscent in previous works such as [9] and more recently [8]. They study finite-horizon trading models with a pre-trade information acquisition stage and find that multiple equilibrium in information acquisition can arise. A crucial feature of these works is that, although the financial market is dynamic, the information market remains static. And it is not clear whether incorporating dynamic information acquisition would still preserve multiplicity. [5] This paper illustrates that dynamic information acquisition

[5] In a previous version of the paper, [10], I show that a finite-horizon repetition of [1] economy has a unique equilibrium, despite that both the financial market and information market are dynamic.

works like bubble: self-fulfilling beliefs about future generations' information choice arising in an infinite-horizon framework seem critical in generating multiplicity.

Moreover, this paper explores how economic primitives affects the dynamic complementarity and offers new implications. In particular, it finds that there is a nontrivial relationship between the persistence of stock fundamental and the strength of dynamic complementarity. This has not been explored in previous works which assume that the stock fundamental is time-invariant. There are also interesting interactions between the *financial market* equilibrium and *information market* equilibrium: at the high-volatility financial market equilibrium, higher fundamental persistence always strengthens the dynamic complementarity; whereas at the low-volatility financial market equilibrium the inverse might be true, and there may exist a U-shaped relationship where the dynamic complementarity is most prominent with intermediate level of fundamental persistence.

The theory is also related to the literature that studies exogenous asymmetric information trading models in an infinite horizon, pioneered by [11, 2] and [12]. It is particularly related to models that study overlapping generations of investors [5, 6, 3, 7, 4][6] . Although the physical structure of my paper is very close to these papers, in my model the information acquisition choice is endogenous. [13] study a dynamic overlapping-generations model with private information where, similar to this paper, a dynamic informational linkage is present: information gets incorporated into the price only if informed traders expect future traders to also impound their information in the price. Unlike this paper, however, it does not concern the issue of multiplicity.[7]

1.2 Model Economy

Time is discrete and runs from $-\infty$ to $+\infty$. The economy is populated by a continuum of overlapping generations risk-averse agents who consume a single consumption good. The good is treated as the numeraire. There are two assets in the economy: a bond in perfect elastic supply, paying a return R;[8] and a stock that pays a dividend

$$D_t = F_t + \varepsilon_t^D \tag{1.1}$$

[6] This literature identifies high volatility equilibria and low volatility equilibria with different stock price sensitivity with respect to noise trader risks.

[7] I thank an anonymous referee for pointing this out.

[8] Alternatively one can interpret the bond as a storage technology without nonnegative constraint.

each period. F_t is the persistent component of the dividend process. Later I call F_t the stock fundamental. The stock fundamental follows an AR(1) process:

$$F_t = \rho^F F_{t-1} + \varepsilon_t^F, 0 \leq \rho^F \leq 1. \tag{1.2}$$

The stock supply, x_t, follows an AR(1) process as well:

$$x_t = \rho^x x_{t-1} + \varepsilon_t^x, 0 \leq \rho^x \leq 1. \tag{1.3}$$

As in [2], I assume that there is a public signal every period about the current fundamental:

$$S_t = F_t + \varepsilon_t^S. \tag{1.4}$$

The shock vector $\boldsymbol{\varepsilon_t} = [\varepsilon_t^D, \varepsilon_t^F, \varepsilon_t^x, \varepsilon_t^S]$ is i.i.d. over time, with mean 0 and covariance matrix $diag(\sigma_D^2, \sigma_F^2, \sigma_x^2, \sigma_S^2)$.

Investors live for two periods. When they are born, they are endowed with a certain amount of wealth and also observe the entire history of the dividend and stock price. They are then offered an opportunity to acquire information at some cost χ. If they choose to acquire information, they also observe the history of the stock fundamental. I call investors who choose to acquire information the "informed" investors and the rest "uninformed." The information set of the generation-t uninformed is

$$\Omega_t^U = \{P_s, D_s, S_s\}_{s=-\infty}^t,$$

and that for the informed is

$$\Omega_t^I = \{P_s, D_s, S_s, F_s\}_{s=-\infty}^t.$$

As is standard in this class of models, an informed investor, observing the history of the fundamental and stock price, can perfectly deduce the stock supply. For uninformed investors, their conditional expectations are derived from Kalman filter equations. We use $\hat{F}$ and $\hat{x}$ to denote the conditional mean of the current fundamental and stock supply for the uninformed:

$$\hat{F}_t = E(F_t|\Omega_t^U) \tag{1.5}$$

$$\hat{x}_t = E(x_t|\Omega_t^U). \tag{1.6}$$

After the information acquisition stage, the financial market opens and trade occurs. After that, old investors exit and consume their wealth. The timeline is summarized in figure 1.1.

The individual agents' problem is as follows. First, they make their information acquisition choice:

$$V = \max\{V^I, V^U\},$$

where V^I denotes the expected utility of generation-t informed investors, and V^U denotes the expected utility for the generation-t uninformed. V^I and V^U are in turn determined by agents' portfolio choice:

$$\begin{aligned}
V^i &= \textstyle\int_P V^i(P) dH_1(P) \\
V^i(P) &= \max_{e,b,c'} \textstyle\int_{P',F',\boldsymbol{\varepsilon}'} U(c') dH_2(P', F', \boldsymbol{\varepsilon}' | \Omega^i) \\
eP + b &\leq w - 1\{i = I\}\chi \\
c' &\leq (D(F', \boldsymbol{\varepsilon}') + P')e + Rb,
\end{aligned}$$

where $U(c) = -\exp(-\alpha c)$, α is the risk-averse parameter. $D(F, \boldsymbol{\varepsilon}) = F + \varepsilon^D$, H_1, H_2 are determined in general equilibrium.

1.2.1 Equilibrium Definition

As is standard in the literature, I will focus on the stationary equilibrium.

Definition 1.2.1 *Denote $s = \{\hat{F}, F, x\}$. A steady state is $\{P(s), \lambda, \{e_i(s), b_i(s)\}_{i=U,I}\}$ s.t.*

1. *$e_i(s), b_i(s)$ solves the uninformed and informed agents' problem given $P(s)$.*
2. *The market clears: $\lambda e_I(s) + (1 - \lambda)e_U(s) = x(s), \forall s_t, t$.*
3. *$V_U = V_I$ if $\lambda \in (0, 1)$; if $\lambda = 0$, $V_U \geq V_I$; if $\lambda = 1$, $V_U \leq V_I$,*

where $\hat{F}$ is the conditional expectation given by equation 1.5. The last condition guarantees that agents' information choice is optimal. For instance, if there is a positive fraction of both informed and uninformed investors ($\lambda \in (0, 1)$), it has to be the case that the expected utility of the informed and the expected utility of the uninformed are equalized.

It is challenging to solve a noisy rational expectations Model with general, potentially nonlinear, price functions. Hence, in later analysis, I accord with the literature and restrict my attention to the class of linear equilibrium. I conjecture that the price function takes the following form:

Definition 1.2.2 *A linear equilibrium is a steady state where price functions are linear with respect to their arguments. That is, there exists* $\{a, p_{\hat{F}}, p_F, p_x\}$ *such that*

$$P(s) = a + p_{\hat{F}}\hat{F} + p_F F - p_x x. \tag{1.7}$$

Except as otherwise noted, in later sections I will restrict attention to linear equilibrium.

1.3 Multiplicity in Information Acquisition

The purpose of this section is to formally establish that there is multiplicity in information acquisition in this economy (theorem 1). To do so, I take the following steps:

1. First define steady states where information (i.e. the fraction of informed investors λ) is exogenous. Denote it as the exogenous-information steady state $\Phi(\lambda)$ (definition 1.3.1).
2. At each $\Phi(\lambda)$ compute the difference in the expected utility of the informed and the expected utility of the uninformed. Denote it as the value of information $\pi(\lambda)$ (definition 1.3.2).
3. If $\pi(\lambda)$ is equal to some measure of the utility cost of acquiring information (unless at boundary), $\Phi(\lambda)$ is a steady state (lemma 1.3.2).
4. If there are multiple values of λ, we find multiple steady states (lemma 1.3.3 and theorem 1).

Step 1: Define the exogenous-information steady state $\Phi(\lambda)$

Intuitively, the auxiliary concept of the exogenous-information steady state is just a steady state where all generations of investors' information are *exogenously* fixed, as studied in [5], [3], and [7]:

Definition 1.3.1 *An exogenous-information steady state given* λ *is* $\Phi(\lambda) = \{P(s), \lambda, \{s_i(s), b_i(s\}_{i=U,I}\}$ *such that it satisfies condition 1 and 2 in definition 1.2.1.*

As is well known in this literature, there exists multiple exogenous-information linear steady states given any λ. To make the exposition transparent, I will focus on the *high-volatility* equilibrium in the main part of the analysis[9] . Note that I do not take a stand on which equilibria one *should* select, as both low-volatility and high-volatility equilibria have desirable properties. The purpose of focusing on the high-volatility equilibrium is to illustrate that there is a new source of multiplicity associated with agents' information choice.

[9] The insight obtained in the main part of the paper carries over to low-volatility equilibria as well, as shown in the robustness check section.

One may wonder about the existence of the exogenous-information steady state. As in [5], I provide conditions such that there exists a unique exogenous-information steady state at least locally near $\lambda = 0$.

Assumption 1

$$(R-\rho^x)^2 - 4\alpha^2\sigma_x^2\left(\left(1+\frac{\rho^F}{R-\rho^F}(\theta_2+\theta_3)\right)^2\left[\left(\rho^F\right)^2\Sigma_0+\sigma_F^2\right]+\left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2+\left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2\right)>0,$$

where Σ_0 is given by A.26, θ_2 is given by A.28, and θ_3 is given by A.29.

Lemma 1.3.1 *Under assumption 1, a high-volatility exogenous-information steady state $\Phi(\lambda)$ exists and is unique for λ sufficiently close to 0.*

Proof. See the appendix. ■

In later analysis, I assume that assumption 1 holds.

Step 2: Define the value of information $\pi(\lambda)$

The value of information is the ratio of the expected utilities of the informed and uninformed *net of information cost* at each high-volatility exogenous-information steady state indexed by λ.

Definition 1.3.2 *Denote the expected utility of the informed W^I and uninformed W^U at each $\Phi(\lambda)$. Define the following*

$$\pi(\lambda) = W^U/W^I,$$

where $W^i, i = I, U$ are given by

$$\begin{aligned} W^i &= \textstyle\int_P W^i(P)dF_t(P) \\ W^i(P) &= \max_{e,b,c'} \textstyle\int_{P',F',\boldsymbol{\varepsilon}'} U(c')dH(P',F',\boldsymbol{\varepsilon}'|\Omega^i) \\ eP+b &\leq w \\ c' &\leq (D(F',\boldsymbol{\varepsilon}')+P')e+Rb. \end{aligned}$$

Step 3: Compare the value of information $\pi(\lambda)$ with some measure of the information cost The next lemma shows that the value of information function $\pi(\lambda)$ allows us to directly compare the expected gain from acquiring information with the cost of acquiring information and determine whether $\phi(\lambda)$ is a steady state.

Lemma 1.3.2 $\forall\lambda \in (0,1)$, *$\Phi(\lambda)$ is a steady state if and only if*

$$\pi(\lambda) = \exp(\alpha R\chi).$$

For $\lambda = 0$ (1), $\Phi(\lambda)$ is a steady state if and only if

$$\pi(\lambda) \leq (\geq) \exp(\alpha R\chi).$$

Proof. Choose the case $\lambda \in (0,1)$. Verifying the other cases is similar. It can be shown that under Constant absolute risk aversion utility, $V^U = W^U$; $V^I = W^I e^{\alpha R \chi}$. Thus, if $\pi(\lambda) = e^{\alpha R \chi}$ holds, then

$$\frac{V^U}{V^I} = \frac{W^U}{W^U e^{\alpha R \chi}} = \frac{\pi(\lambda)}{e^{\alpha R \chi}} = 1.$$

Thus all of the conditions for a steady state hold for $\Phi(\lambda)$, and $\Phi(\lambda)$ is a steady state. ■

Step 4: Prove multiplicity

The goal of the last step is to show that (under some conditions) there exists multiple values of λ satisfying the conditions stated in lemma 1.3.2.

To do so, let me state the following lemma:

Lemma 1.3.3 *Suppose*

$$(1-\theta_0)\rho^F > \rho^x + \phi \tag{1.8}$$

for some $\theta_0 \in [0,1]$ given by A.30 and ϕ given by **??***, then*

$$\frac{d\pi(\lambda)}{d\lambda} > 0 \text{ for } \lambda \text{ sufficiently small.}$$

$\theta_0, \rho^F, \rho^x, \phi$ are either structural parameters or functions of the structural parameters. the lemma says that under certain conditions (which we will elaborate in the next section), the incentive of people to become informed, $\pi(\lambda)$, increases as there are more informed investors. This is in sharp contrast with the classical substitution effect in [1].

The next theorem establishes the multiplicity result:

Theorem 1 *Under condition 1.8, there exits χ such that multiple steady states exist.*

Proof. The proof follows from the intermediate value theorem and lemma 1.3.3. The value of information function $\pi(\lambda)$ is differentiable, and hence continuous. Given that $\pi'(\lambda) > 0$ for λ sufficiently small, choose χ such that $e^{\alpha\chi} = \pi(\lambda_1)$ for some λ_1 sufficiently small but strictly positive. Then we know that λ_1 is a steady state. Also, we know that $\pi(0) < \pi(\lambda_1) = e^{\alpha\chi}$. Thus, we know that $\lambda_0 = 0$ is another steady state because no one is informed and the gain from acquiring information is less than the cost. Thus there are at least two steady states (see figure 1.2). ■

1.4 Intuition and a Heuristic Proof of Lemma 1.3.3

The key step toward multiplicity is lemma 1.3.3. A rigorous proof is delegated to the appendix. This section is devoted to explaining the main steps and intuition for why the value of information is locally increasing ($\pi'(\lambda) > 0$) for sufficiently small λ.

To begin, as in [1], the value of information is given by the ratio of the stock payoff uncertainty faced by the uninformed and informed. More precisely, the value of information is given by

$$\pi = \sqrt{\frac{Var(D' + P'|\Omega^U)}{Var(D' + P'|\Omega^I)}},$$

where D' is the next period dividend and P' is the next period stock price.

In this heuristic proof, instead of focusing on the ratio of uncertainty, we proxy the value of information with the *difference* in the conditional variance of the stock payoffs between the uninformed and informed:

$$\Delta V := Var(D' + P'|\Omega^U) - Var(D' + P'|\Omega^I).$$

To derive an expression for ΔV, plug equation 1.1 and equation 1.7 into the stock payoff $D' + P'$:

$$\begin{aligned} D' + P' &= \underbrace{F' + \varepsilon'^D}_{D'} + \underbrace{a + p_{\hat{F}}\hat{F}' + p_F F' - p_x x'}_{P'} \\ &= a + (1 + p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon'^D. \end{aligned} \tag{1.9}$$

Then plug in the law of motion of F' (equation 1.2), the law of motion of x' (equation 1.3), and the law of motion of $\hat{F}'$ (equation A.5). Then rearranging, one obtains

$$\begin{aligned} D' + P' &= a + (1 + p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon'^D \\ &= \underbrace{a + e_1\hat{F} + e_2\hat{x}}_{\text{common knowledge}} + \underbrace{e_3 F - e_4 x}_{\text{known to the informed}} + \underbrace{e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime} + e_8\varepsilon^{S\prime}}_{\text{shock}}, \end{aligned} \tag{1.10}$$

where coefficients e_i are given in A.6 through A.13 and are all positive. $\hat{F} = E(F|\Omega^U)$ and $\hat{x} = E(x|\Omega^U)$ are the uninformed's estimate of the current period fundamental and the stock supply respectively. As shown in expression 2.9, we can decompose the equation into three components. The first component, $a + e_1\hat{F} + e_2\hat{x}$, is common knowledge. The third component, $e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime} + e_8\varepsilon^{S\prime}$, consists of future shocks that no one could possibly know today. The second component, however, is only known to the informed. Thus, the difference in uncertainty between informed and uninformed is just the conditional volatility of the second component:

$$\begin{aligned} \Delta V &= Var(D' + P'|\Omega^U) - Var(D' + P'|\Omega^I) \\ &= Var(e_3 F - e_4 x|\Omega^U) \\ &= e_3^2 Var(F|\Omega^U) + e_4^2 Var(x|\Omega^U) - 2e_3 e_4 Cov(F, x|\Omega^U). \end{aligned} \tag{1.11}$$

The first two variance terms reflect, respectively, the predictive role of the fundamental and supply. The last term is negative, reflecting the fact that conditional on observing the current stock price, fundamental and supply are correlated and the bigger the correlation, the lower the value of information. Taking derivatives with respect to each term, we have

$$\frac{\partial \Delta V}{\partial \lambda} = \frac{\partial e_3^2 Var(F|\Omega^U)}{\partial \lambda} + \frac{\partial e_4^2 Var(x|\Omega^U)}{\partial \lambda} - \frac{\partial 2 e_3 e_4 Cov(F,x|\Omega^U)}{\partial \lambda}. \tag{1.12}$$

Thus, whether ΔV is locally increasing in λ depends on three terms. Next I will examine the value of each term, taking λ very close to zero.

1.4.1 The predictive role of fundamental F

The first term $\partial e_3^2 Var(F|\Omega^U)/\partial\lambda$ in equation 1.12 reflects how perturbations in λ affect the value of information through the predictive role of fundamental F. When the share of informed, λ, increases, two opposing forces affect the value of $e_3^2 Var(F|\Omega^U)$. On the one hand, there is classic substitutability: that there are more informed investors today implies a more informative current stock price. Thus, the conditional variance of fundamental $Var(F|\Omega^U)$ is reduced. On the other hand, since there are more informed investors in the future, the future stock price loads more heavily on the fundamental, and thus the loading coefficient e_3 increases.

To compare the two forces, the crucial observation is that classical substitutability is absent when $\lambda \to 0$. More precisely:

$$\lim_{\lambda \to 0} \frac{\partial Var(F|\Omega^U)}{\partial \lambda} = 0. \tag{1.13}$$

This is due to the nature of the Kalman filter: $Var(F|\Omega^U)$ depends on p_F and p_x only to the extent that it depends on $(\dfrac{p_F}{p_x})^2$. This implies that the derivative of $Var(F|\Omega^U)$ with respect to λ is proportional to $\frac{p_F}{p_x}$. Also note that when $\lambda \to 0$, $p_F \to 0$ because there are no informed investors that knows perfectly the value of F. Therefore,

$$\frac{\partial Var(F|\Omega^U)}{\partial \lambda} \propto \frac{p_F}{p_x} \to 0 \text{ when } \lambda \to 0.$$

Thus, perturbing λ near $\lambda = 0$ does not affect the conditional uncertainty about the current fundamental faced by the uninformed.

On the other hand, the loading coefficient e_3^2 is *strictly* increasing in λ. To see this, we need to derive an expression for e_3. For simplicity, ignore the law of motion for $\hat{F}'$ and just plug in the law of motion for F' and x' into equation 2.9:

$$\begin{aligned} D' + P' &= a + (1+p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon'^D \\ &= a + (1+p_F)(\rho^F F + \varepsilon^{F\prime}) - p_x(\rho^x x + \varepsilon^{x\prime}) + p_{\hat{F}}\hat{F}' \\ &= a + (1+p_F)\rho^F F - p_x \rho^x x + p_{\hat{F}}\hat{F}' + (1+p_F)\varepsilon^{F\prime} - p_x \varepsilon^{x\prime}. \end{aligned} \tag{1.14}$$

Thus, e_3 and e_4 are approximately

$$e_3 \approx \rho^F (1+p_F) \tag{1.15}$$

$$e_4 \approx \rho^x p_x. \tag{1.16}$$

As $\lambda \to 0$, the loading of stock price on the fundamental, p_F, converges to 0, but the loading of the stock *payoffs* on the fundamental, e_3, converges to some strictly positive number ρ^F. This is because of the presence of the interim dividend payout, which introduces additional sensitivity into the future stock payoff compared with the current stock price (see the second term in equation 1.9). Thus, the derivative of e_3^2 with respect to λ is proportional to $2e_3$, which in turn is roughly equal to $2\rho^F(1+p_F)$ which converges to some strictly positive number when λ is sufficiently small:

$$\frac{\partial e_3^2}{\partial \lambda} > 0 \text{ when } \lambda \to 0. \tag{1.17}$$

Combining the static substitutability (equation 1.13) and the dynamic complementarity (equation 1.17), one can show that the first term in equation 1.12 is always positive at the limit:

$$\lim_{\lambda\to 0}\frac{\partial[e_3^2 Var(F|\Omega^U)]}{\partial\lambda} = \underbrace{\lim_{\lambda\to 0}\underbrace{\frac{\partial e_3^2}{\partial\lambda}}_{>0}\underbrace{Var(F|\Omega^U)}_{>0}}_{\text{dynamic complementarity}>0} + \underbrace{\lim_{\lambda\to 0}\underbrace{\frac{\partial Var(F|\Omega^U)}{\partial\lambda}}_{=0} e_3^2}_{\text{static substitutability}=0} > 0$$

1.4.2 The predictive role of supply x

The second term $\partial e_4^2 Var(x|\Omega^U)/\partial\lambda$ in equation 1.12 captures the predictive role of supply x. As discussed in [8], the equilibrium price becomes a noisier signal of supply when there are more informed investors. This force tends to increase the conditional supply uncertainty faced by the agents and thus increase the value of information. More precisely, one can show that

$$Var(x|\Omega^U) = (\frac{p_F}{p_x})^2 Var(F|\Omega^U).$$

As λ increases, the stock price becomes more sensitive to the fundamental, and thus the ratio $\frac{p_F}{p_x}$ increases. This tends to push up the conditional uncertainty of supply and thus increases the value of acquiring information and leads to multiplicity.

In this model, however, this effect is absent locally around $\lambda = 0$. Again, this result follows from the nature of the Kalman filter: like $Var(F|\Omega^U)$, $Var(x|\Omega^U)$ is also an (implicit) function of $(\frac{p_F}{p_x})^2$. Thus the derivative of $Var(x|\Omega^U)$ with respect to λ is proportional to $\frac{p_F}{p_x}$. As a result, it tends toward zero as λ tends toward zero:

$$\frac{\partial Var(x|\Omega^U)}{\partial\lambda} \propto \frac{p_F}{p_x} \to 0, \text{as } \lambda \to 0.$$

In fact, one can further strengthen the statement by observing that when $\lambda \to 0$, the stock price becomes a perfect signal of supply. This implies that there is no uncertainty about the stock supply locally around $\lambda = 0$. That is $\lim_{\lambda \to 0} Var(x|\Omega^U) = 0$. To sum up

$$\lim_{\lambda \to 0} \frac{\partial [e_4^2 Var(x|\Omega^U)]}{\partial \lambda} = \underbrace{\lim_{\lambda \to 0} \frac{\partial e_4^2}{\partial \lambda} \underbrace{Var(x|\Omega^U)}_{=0}}_{=0} + \underbrace{\lim_{\lambda \to 0} \underbrace{\frac{\partial Var(x|\Omega^U)}{\partial \lambda}}_{=0} e_4^2}_{=0} = 0 \tag{1.18}$$

Thus, in this model, the supply channel does not play any role locally around $\lambda = 0$. This is different from the work by [8] that emphasizes the supply channel. We can safely ignore this term henceforth.

1.4.3 Correlation between the fundamental and supply as an offsetting force

The third term $-\partial e_3 e_4 Cov(F, x|\Omega^U)/\partial\lambda$ in equation 1.12 reflects the fact that an increase in λ may reduce the value of information through the correlation channel. The logic is as follows. When λ increases from zero to some strictly positive increment, the price becomes a noisier signal of stock supply. This increases the standard deviation of the stock supply relative to that of the stock fundamental and thus the conditional correlation between the fundamental and supply. When the correlation increases, information about the fundamental is not that valuable because any signal that predicts a good fundamental also predicts a large stock supply. The two forces cancel out each other, making the signal less useful in predicting the future stock payoff.

To derive the sign of $\frac{\partial \Delta V}{\partial \lambda}$, we need to compare the third term with the first term (recall that the second term vanishes as λ tends to zero). The rough intuition is the following. As can be seen in equation 1.15, the loading of the future stock payoffs on the current fundamental, e_3, is proportional to ρ^F. Thus, e_3^2 is proportional to $(\rho^F)^2$. When taking derivatives in equation 1.12, we can factor out the constants and thus the first term, $\partial e_3^2 Var(F|\Omega^U)/\partial\lambda$, is proportional to $(\rho^F)^2$. Similarly, the third term,$-\partial e_3 e_4 Cov(F, x|\Omega^U)/\partial\lambda$, is proportional to $\rho^F \rho^x$. Therefore, roughly speaking, for the first term to dominate the third term we need $(\rho^F)^2$ to be greater than $\rho^F \rho^x$, or equivalently, ρ^F greater than ρ^x.

When the exact formulas of e_3 and e_4 are plugged in, one can formally show that when $\lambda \to 0$,

$$\frac{\partial \Delta V}{\partial \lambda} > 0 \iff (1 - \theta_0)\rho^F > \rho^x, \tag{1.19}$$

where the expression of θ_0 is given in A.30 and is always between 0 and 1. Thus, for ΔV to be locally increasing in λ, it is necessary that the fundamental is sufficiently more persistent than supply.

The $1 - \theta_0$ term measures the *information advantage* of the informed at $\lambda = 0$[10] . It is always nonnegative because informed investors are always at a (weak) information advantage

[10] Formally, it captures how sensitive informed agents' estimate of the fundamental is with respect

relative to the uninformed. The presence of $1-\theta_0$ in condition 1.19 says that multiplicity is more likely to arise when the information advantage of the informed is larger. The intuition is as follows. Whether multiplicity arises depends on the strength of the dynamic complementarity, or how much more sensitive the future stock price would become with respect to the fundamental upon a marginal increase in λ. When the information advantage of the informed is very large, the informed's demand is much more sensitive to the fundamental than the uninformed. Thus, a marginal increase in the share of the informed tomorrow makes tomorrow's aggregate demand, and hence tomorrow's stock price, much more sensitive to the fundamental. This implies that the dynamic complementarity is stronger, and thus multiplicity is more likely to arise.

So far we have used ΔV as a proxy for the value of information and illustrate a necessary and sufficient condition for multiplicity to arise (condition 1.19). In the rigorous proof, there is an additional term ϕ in condition 1.8. This ϕ term captures the level effect of changing λ. Namely, perturbing λ changes not only the *difference* in uncertainty but also the *level* of uncertainty. This has a nontrivial effect on the value of information. To see this,

$$\begin{aligned}\pi &= \sqrt{\tfrac{Var(D'+P'|\Omega^U)}{Var(D'+P'|\Omega^I)}} \\ &= \sqrt{1+\tfrac{Var(D'+P'|\Omega^U)-Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^I)}} \\ &= \sqrt{1+\tfrac{\Delta V}{Var(D'+P'|\Omega^I)}}.\end{aligned} \tag{1.20}$$

Thus, the value of information is a monotonic function of $\frac{\Delta V}{Var(D'+P'|\Omega^I)}$. Not only the difference but also the level of uncertainty $Var(D'+P'|\Omega^I)$ enter into the expression of π and matter. The term ϕ is complicated and hard to characterize analytically. Yet, as shown in figure 1.3, where I numerically solve and plot the parameter pair (ρ^F, ρ^x) that satisfies condition 1.8, the general insight that the fundamental should be more persistent carries over.

1.5 Comparative Statics

So far we have illustrated that for the multiplicity in information acquisition to arise, it is crucial that the fundamental is more persistent than supply. This section investigates how other parameters, $(\sigma_F^2, \sigma_D^2, \sigma_x^2, R, \alpha)$ and most crucially σ_S^2, affect the multiplicity result.

to the true fundamental relative to the uninformed's:

$$1-\theta_0 = \lim_{\lambda\to 0} \frac{\partial[E(F|\Omega^I)-E(F|\Omega^U)]}{\partial F}.$$

As shown in figure 1.4, increasing the precision of the public signal shrinks the multiplicity region. This result can be understood as follows. As discussed before, $1 - \theta_0$ captures the information advantage of the informed. When the public signal becomes more precise, this information advantage vanishes. This makes condition 1.8 harder to satisfy. Thus, the multiplicity region shrinks.

This result provides an interesting perspective on recent policy attempts to provide more precise public information. It says, for a regulator seeking to stabilize asset prices, it is desirable to disclose more precise public information because it helps to eliminate equilibrium multiplicity.

Other comparative statics exercises are collected in figure 1.5. All the results can be understood by examining how these parameters affect the value of θ_0 and hence condition 1.19. First, reducing the dividend noise σ_D^2 shrinks the multiplicity region, since the dividend becomes a more precise signal of the fundamental and therefore θ_0 increases. Likewise, a decrease of σ_F^2 reduces prior uncertainty and therefore makes uninformed agents rely less upon dividend information. This reduces the sensitivity parameter θ_0. Thus, the multiplicity region expands. Last, varying σ_x^2 hardly affects the multiplicity region because the price signal contains pure noise when there are no informed agents. Therefore, price signals drop out of uninformed agents' Bayesian updating problem for the fundamental (although it is still useful in predicting future supply). Thus, varying σ_x^2 does not change the value of θ_0. Thus, the multiplicity region hardly changes.[11] Similarly, risk averse coefficient α and interest rate parameter R do not affect multiplicity region, because they do not enter into the expression of θ_0.

Low Volatility Equilibria

In the main part of the paper, I focus on high-volatility equilibria to illustrate the multiplicity in information acquisition. This is convenient because the ϕ term becomes very small with high-volatility equilibria.[12] This section checks whether the main results of the paper extend to the low-volatility equilibria. The answer is yes. To illustrate, I set the parameters to the benchmark level and plot the multiplicity region corresponding to high- and low-volatility equilibria, respectively. As can be seen in figure 1.6, the multiplicity region (surrounded by dashed green lines) shrinks with low-volatility equilibria. Note that the multiplicity region displays backward bending with low-volatility equilibrium, due to the *level* effect captured by the ϕ term in condition 1.8. Specifically, it is possible that an increase in the share of the informed induces an increase in the loading of the equilibrium price on supply p_x, which leads to

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[11] Varying σ_x^2 does affect the multiplicity region through the feasibility condition (equation 1). This effect is present only when the volatility parameters are set large. This case is checked in the robustness check section.

[12] To understand this, note that ϕ is proportional to $1/V^U$. In the high-volatility equilibrium, $V^U = Var(P' + D'|\Omega^U)$ is generally very big. Thus, $\phi \approx 0$.

higher uncertainty and thus, by equation 1.20, lower value of information. When the persistence parameter ρ^F gets bigger, this level effect becomes more powerful due to a stronger dynamic linkage.

Large volatility parameters

In the main part of the paper, the volatility parameters are set to be small ($\sigma_F^2 = \sigma_D^2 = \sigma_x^2 = \sigma_S^2 = 0.01$). Small volatility is convenient because the feasibility constraint (assumption 1) never binds for any feasible $(\rho^F, \rho^x) \in [0,1] \times [0,1]$. This section shows what the multiplicity region looks like when the volatility parameters are set to be large.

When the volatility parameters are set to be large, the feasibility constraint (assumption 1) kicks in, which affects the multiplicity region. However, the general conclusion that the fundamental needs to be more persistent than supply for multiplicity to arise is robust. To illustrate, I set volatility parameters to 10 one by one and plot the resulting new multiplicity region (figure 1.7). The multiplicity region is surrounded by a red dashed line. The feasibility constraint becomes most stringent when σ_F^2 and σ_x^2 are set to be large. This is in line with the literature studying overlapping-generations asymmetric information trading models, which typically set σ_x^2 to be a very small number.

Last, it is natural to ask how robust the multiplicity result is when agents live for more than two periods. This question, however, cannot be addressed within the current framework because when agents live for more than two periods, their incentives to acquire information depends on the conditional mean of stock fundamental. The conditional mean, however, is affected by random shocks (see equation A.5). This in turn implies that agents' incentives to acquire information become functions of random shocks, as do the price coefficients. With random price coefficients, the equilibrium price is no longer normally distributed. This breaks down the classic linear-normal framework and calls for an alternative approach.[13]

1.6 Conclusion

This paper studies implications of dynamic information acquisition in an otherwise standard infinite-horizon asymmetric information trading model. It is shown that multiplicity in information acquisition could arise in such an environment. This finding is due to the dynamic complementarity in information acquisition: current investors have more incentive to become informed if more investors are informed in the future. The dynamic complementarity dominates

[13] [8] studies information acquisition with long-lived agents. There information market only opens at the beginning of the economy. Crucially, the prior mean is normalized to be zero. Thus, the model remains tractable.

classical static substitutability ([1]) and leads to multiplicity. The interim dividend payout is important because it introduces additional sensitivity with respect to the fundamental into the future stock payoff compared with the current stock price. It is also crucial that the fundamental is more persistent than the stock supply.[14] The model has some other implications. For example, multiplicity in information acquisition becomes less likely to arise when the public signal becomes more precise. This suggests that for a regulator seeking to stabilize asset prices, disclosing more precise public information is beneficial because it helps to eliminate equilibrium multiplicity.

[14] [2] and [3] set the persistence parameters to be equal. [4] and [7] choose the fundamental to be more persistent than supply.

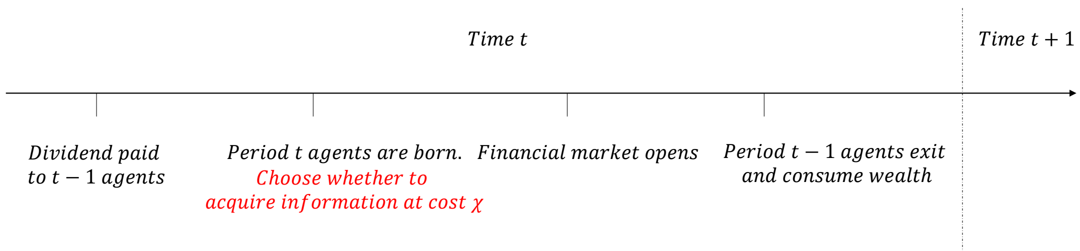
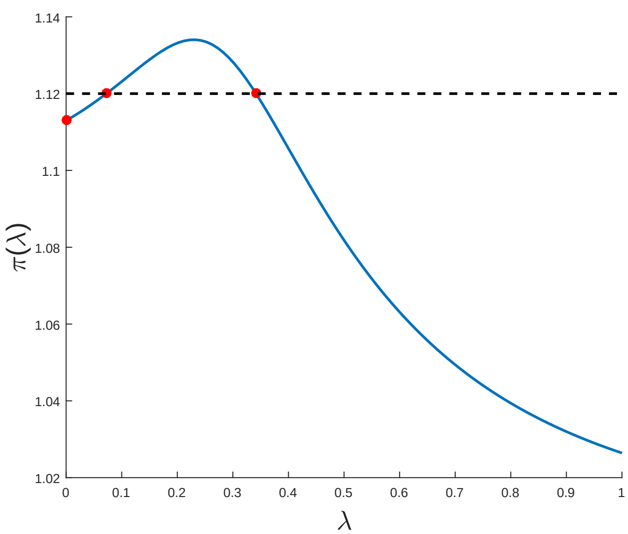
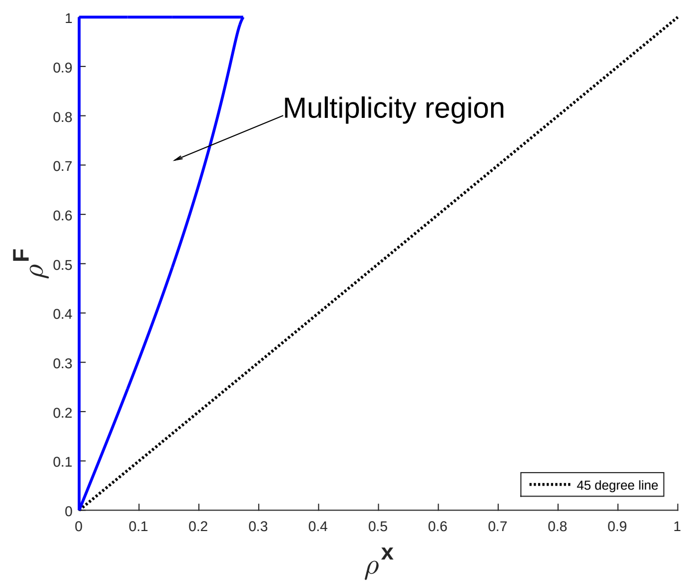


Figure 1.1: Timeline



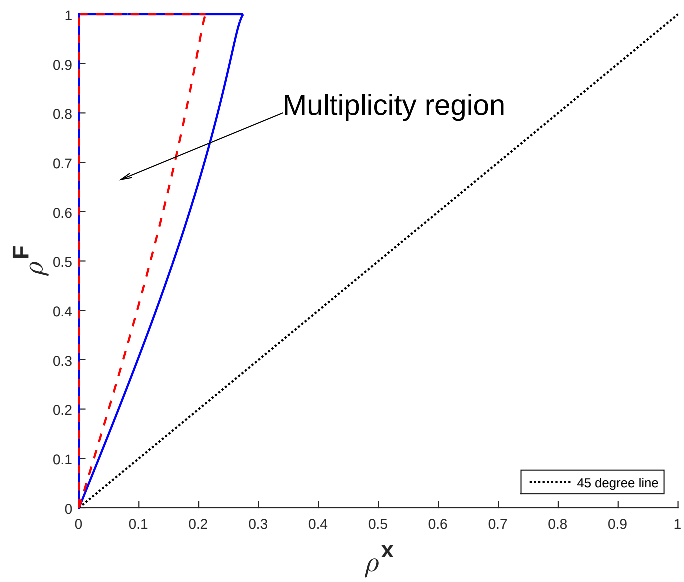
The blue curve depicts numerically solved $\pi(\lambda)$. The black dashed line depicts the information cost. Red dots are numerically solved steady states. Parameter values: $\alpha = 1, R = 1.05, \rho^F = 1, \rho^x = 0, \sigma_F^2 = 0.01, \sigma_D^2 = 0.1, \sigma_x^2 = 0.01.$

Figure 1.2: The value of information $\pi(\lambda)$



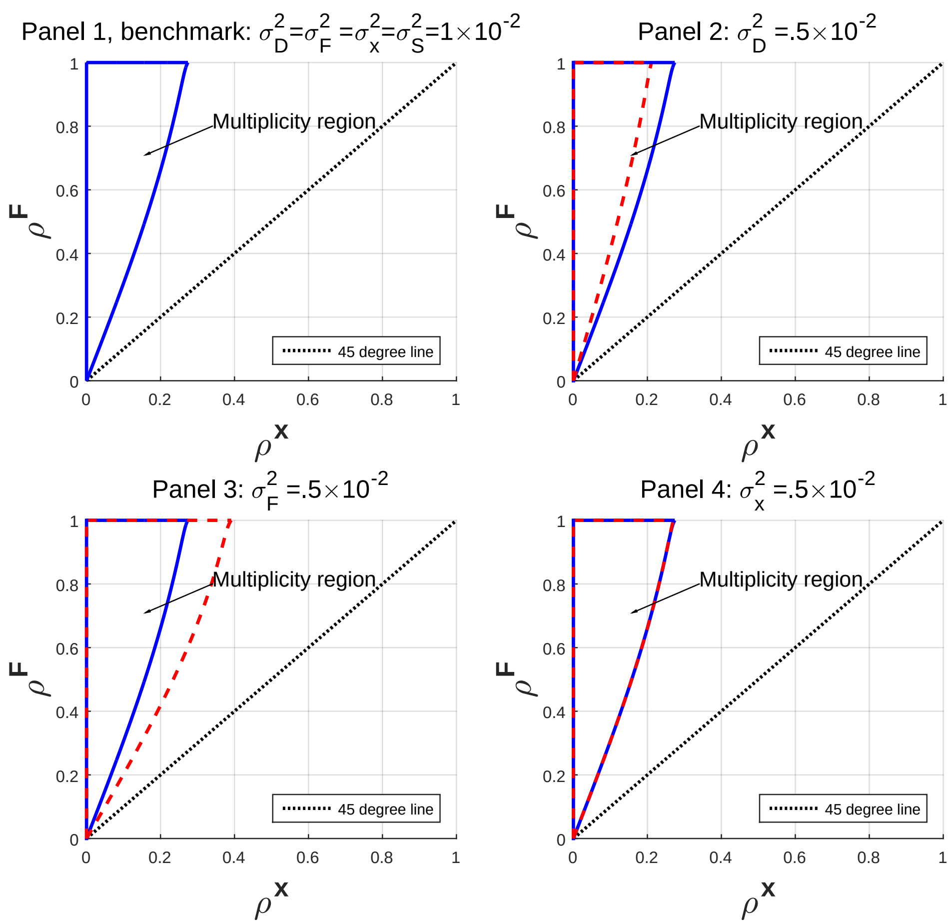
This figure plots the parameter subspace (ρ^x, ρ^F) *where condition 1.8 is satisfied (the area surrounded by the blue curve), and hence multiplicity in information acquisition could arise.*
Parameter values: $R = 1.05, \alpha = 1, \sigma_F^2 = \sigma_D^2 = \sigma_x^2 = \sigma_S^2 = 0.01$

Figure 1.3: Multiplicity Region



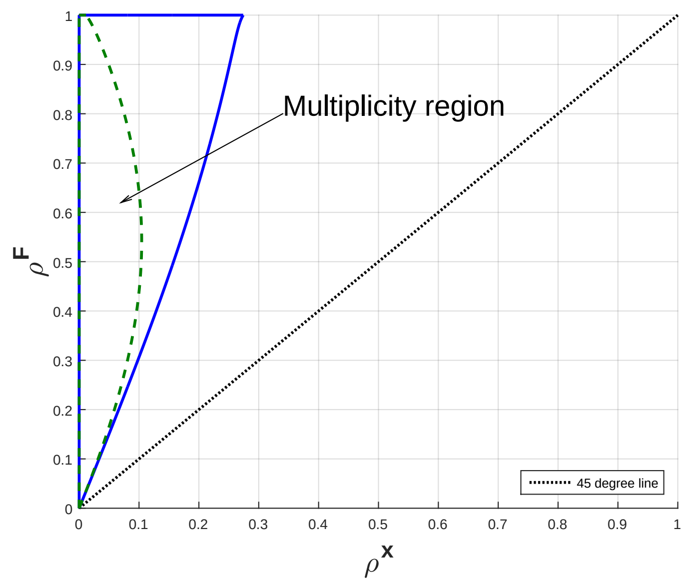
This figure plots how the multiplicity region changes when σ_s^2 decreases from 0.01 to 0.005. The blue area plots the benchmark case whereas the red dashed area plots the case where $\sigma_s^2 = 0.005$.

Figure 1.4: Comparative Statics: $\sigma_s^2 = 0.01 \rightarrow 0.005$



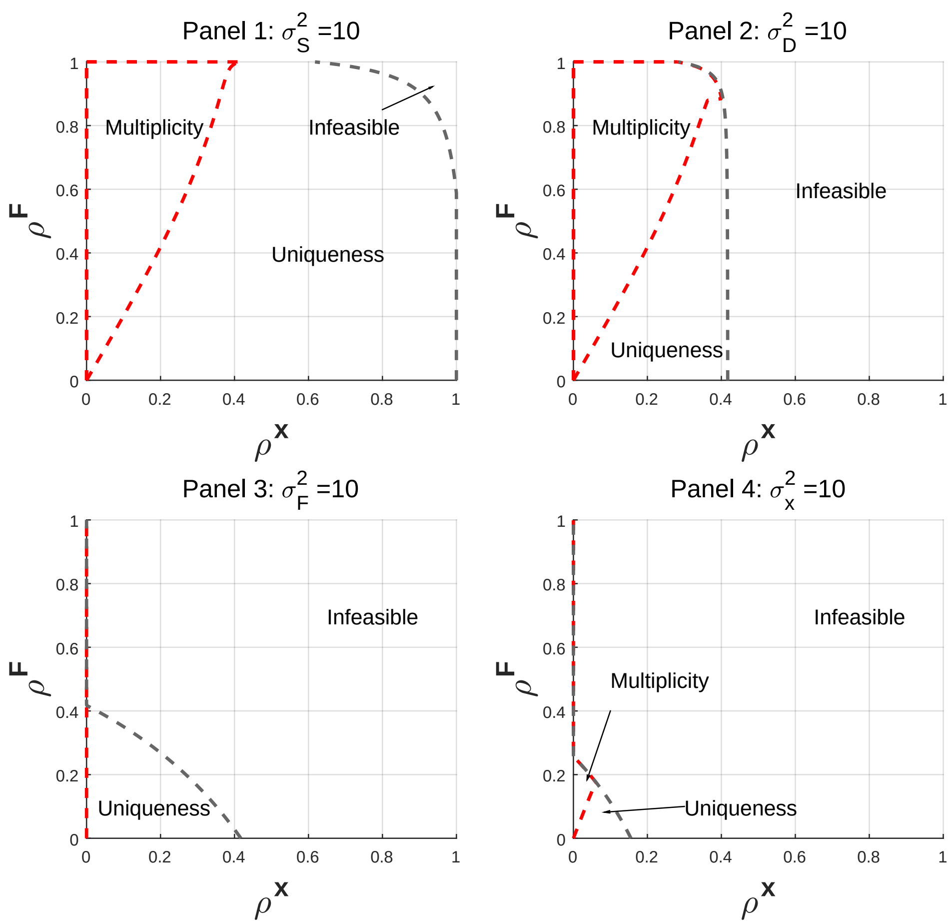
Panels 2-4 plot how the multiplicity region changes when σ_D^2, σ_F^2, and σ_x^2 change from 0.01 *to* 0.005*, respectively. The blue area is the benchmark case. The red areas in panels 2,3 and 4 are the new multiplicity regions when the volatility parameters are set to* 0.005*.*

Figure 1.5: Comparative Statics



The figure plots the multiplicity region under the benchmark parameter values $\sigma_F^2 = \sigma_D^2 = \sigma_x^2 = 0.001$. *The blue area depicts the multiplicity region with high-volatility equilibria. whereas the dashed green area depicts the multiplicity region with low-volatility equilibria.*

Figure 1.6: Robustness Check II: Low-volatility equilibria



Panels 2-4 plots how the multiplicity region changes when σ_D^2, σ_F^2, and σ_x^2 change from 0.001 *to* 10 *respectively. The blue area in panel 1 is the benchmark case where* $\sigma_F^2 = \sigma_D^2 = \sigma_x^2 = 0.001$. *The red area is the new multiplicity region after, say,* σ_F^2 *is changed to* 50. *In panels 2 through 4, all areas to the right of the grey dashed curves violate assumption 1 and are thus infeasible.*

Figure 1.7: Robustness Check I: large volatility parameters

Chapter 2

Land Prices, Collateral Constraints and Secular Stagnation

2.1 Introduction

The Great Recession from 2007-2009 differs considerably from other postwar recessions. First, declines in major macroeconomic variables were huge relative to other postwar recessions. Second, the recovery has been much slower. As shown in Figure 2.1, following an average postwar recession, major macroeconomic variables fully recovered in five years. Yet, the impact of the Great Recession seemed to be much more persistent: GDP and housing price kept declining relative to their respective trends; Labor barely recovered; Investment recovered somewhat but was still 20 percent below trend after five years.[1]

The Great Recession resembles other historical crises such as the Great Depression and Japan's 1990 crisis. Both were severe crises, and were followed by slow recoveries. This suggests a relationship between the *size* of the economic downturn and recovery speed. Why do severe crises tend to be followed by slow recoveries? This paper proposes and quantifies a theory to account for this pattern.

SO

[1] It has been argued that the huge drop of housing price during the recession is largely corrections to the pre-recession housing price boom. Yet despite the pre-recession acceleration, there were still big declines in housing prices relative to its trend after the Great Recession (see Figure B.1 for housing price and Figure B.2 for land price constructed following [14]).

The general view of this paper is that the economy has *multiple regimes*, or locally stable steady states: a positive one with high levels of capital and high land prices, and a negative one with low levels of capital and low land prices. Given multiple isolated steady states, the economy's responses to small and large shocks are different: Small shocks induce local fluctuations around steady states, whereas large shocks trigger transitions between steady states, thus exerting a persistent impact on the economy.

The central piece of the theory is interactions between the households and the firms through the land sector. Specifically, I consider an otherwise standard neoclassical model with a land[2] sector in which land serves dual roles. On the one hand, land serves as *consumption* for households. On the other, it can serve as collateral for firms to finance their borrowing, in particular their *working capital*. In practice, it is common for firms to use their balance sheet assets as collateral to finance their borrowing, including their working capital (See [15, 16]). An important form of collateral assets for U.S. firms is real estate. According to the Flow of Funds, in 2010 real-estate assets made around 50 percent of the nonfinancial assets and 25 percent of the total assets for U.S. nonfinancial corporates. For nonfinancial noncorporates, more than 90 percent of their nonfinancial assets were real-estate.[3]

In this environment, I prove that the law of motion for capital is S-shaped, with possibly multiple locally stable steady states: a "good" one, with high levels of capital and high land prices, and a "bad" one, with low levels of capital and low land prices. The logic is summarized in Figure 2.2. At the good steady state, land prices are high. This implies that firms' working capital constraints are slack, which enhances their employment capacity. As labor and capital are complementary in production, firms have large incentives to invest, which leads to high levels of capital. High levels of capital and high levels of labor together imply high household consumption, and thus strong household demand for land, as the shadow value of wealth is low. This confirms the high land prices in the first place. At the bad steady state, however, land prices are depressed. The low land prices tighten firms' working capital constraints, constraining their employment capacity and their incentives to invest as well. As a result, steady-state levels of capital are low, as is household demand for land, which confirms the initial depressed housing prices.

With multiple locally stable steady states, the model features asymmetric recovery speeds upon shocks of different sizes. Large adverse shocks have a much more persistent impact, as they trigger transitions from one steady state to the other.

The key assumption for multiple steady states to exist is a sufficiently low intratemporal elasticity of substitution between housing and consumption, or equivalently that consumption and

[2] In this paper with slight abuse of language I will use "housing" and "land" interchangeably.

[3] See Federal Reserve Board Z.1 Release, table B.103 and B.104 for nonfinancial corporates and non-corporates respectively.

housing are quite complementary. With low elasticity of substitution, procyclical fluctuations in (nonhousing) consumption expenditures bring about large procyclical fluctuations in aggregate housing prices. The highly volatile housing prices impact the aggregate economy through firms credit constraint, delivering extremely persistent responses upon large transitory shocks. This is in contrast to typical macro models with financial frictions that fail to generate high volatility of asset prices and thus fail to generate strong amplification and persistence quantitatively ([17, 18]).

In the quantitative section of the paper, I calibrate the model to the US economy and quantitatively evaluate the persistence property of the model. To do so, I first calibrate a series of temporary financial shocks to match the observed drop in housing prices since 2007. I then feed this shock, together with the productivity shock from [19] into the model and evaluate how much persistence the model generates. The model is able to generate a slow recovery comparable to the US experience after 2010, and did particularly well in matching the slow recovery of labor. Moreover, it is also able to replicate the sharp and persistent rise of the labor wedge, in particular of its firm component.[4]

In terms of model predictions, as collateral constraints and land price dynamics play central propagation roles in my theory, my model predicts that financial crises with real-estate price busts tend to be followed by slow recoveries. A series of recent empirical papers (e.g. [21], [22], [23]) studying historical cross-country evidence suggest that this is the case. One of their general findings is that historically financial crises are associated with plummeted asset prices (real-estate prices in particular) and are followed by slow recoveries in various macroeconomic variables such as output and labor. In terms of cross-sectional evidence regarding the current recession, [24] documents, using MSA (Metropolitan Statistical Area) level data, that the recovery of local labor market is slower in MSAs with larger housing price declines during the crisis.

The paper is organized as follows. Section 2.2 lays out a baseline model where I prove the main theoretical result of the paper that multiple locally stable steady states exist. Section 2.3 enriches the baseline model with more realistic features and takes it to the data. Section 2.4 concludes.

Literature Review

It is the first paper, to my knowledge, which illustrates that collateral constraints not only amplify shocks, but also generate multiple steady states in a unique equilibrium. Thus it contributes to the vast literature of macro models with collateral constraints, initiated by [25]. [25] considers a model where land serves both as a factor of production and collateral for the firm to

[4] Labor wedge is defined as the log distance between the marginal product of labor and the marginal rate of substitution between consumption and labor. The firm component of the labor wedge is defined, as in [20], the log distance between the marginal production of labor, and the real wage.

finance their investment expenditure. They find that collateral constraints amplify and propagate productivity shocks (and even lead to local indeterminacy[5]) around a unique steady state. Recent papers, such as [15], [16], [26], and []lwz, extend the original [25] framework in various dimensions. Still, all of these papers feature a unique steady state. In this paper, I incorporate two ingredients, *working capital* and the *consumption* role of land, into a canonical macro model with collateral constraints and find that steady-state multiplicity arises. Note that each of the two ingredients alone has already been considered in the literature. And it is the first paper that combines the two elements and systematically investigates its implications.

Specifically, works by [15] and [16] incorporate working capital into macro models with collateral constraints but ignore the consumption role of land. They consider environments in which firms finance their working capital on top of investment expenditures, assuming that physical capital is the only form of asset owned by the firms. Papers by [26] and []lwz consider the consumption role of land, but ignore the working capital aspect.[6] They propose environments in which firms accumulate both physical capital and real-estate assets. Real-estate assets are different from physical capital as they can serve either as consumption goods for households, in addition to as collateral (and a production factor) for the firm. This paper combines elements from both strands of literature and reach a novel conclusion: that multiple locally stable steady states arise.

In addition, the paper is related to the literature that studies the relationship between financial frictions and agents' overborrowing behavior in small open economies (for example [28] and [29]). [30] and [31] demonstrate that collateral constraints introduce aggregate nonconvexities that lead to multiple equilibria under certain parameterizations. Unlike this literature, which focuses exclusively on small open economies without capital accumulation, the focus of this paper is on closed production economies with capital accumulation. Moreover, my model features multiple steady states but a *unique* equilibrium. Thus there is no room in my model for equilibrium multiplicity or sunspots. This is desirable as it frees me from the problem of equilibrium selection.

The paper is also related to an early literature that studies the relationship between capital market imperfections and persistence of initial wealth distribution . [32] illustrates that with non-convex production technology and credit rationing, multiple steady states associated with different distributions of wealth exist. Unlike [32], my mechanism does not rely on a non-convex production technology. [33] illustrates that endogenous interest rate interacts with wealth distribution and leads to multiple steady states when credit rationing presents. Here my result does not rely on the dynamics of interest rate. Moreover, [33]'s primary interest is in long-run

[5] See footnote 16, [25]. Also see [18].

[6] [27] incorporates working capital requirement as one of their sensitivity checks and they do not characterize conditions under which multiple steady states arise.

growth. Thus it is built on the Solow growth model where households saving behavior is treated *exogenously*[7] . My model, in contrast, is built on a standard neoclassical growth model where households saving behavior is forward looking and determined in equilibrium. This makes my framework more suitable for business cycle analysis, which is the focus of this paper.

The paper is also related to the fast-growing literature that views the slow recovery from the Great Recession as a transition between different "regimes," or steady states. [34] shows that perfect real wage rigidity leads to the existence of a continuum of steady states indexed by capital level. [35] uses global game techniques to show that aggregate demand externalities lead to the existence of multiple locally stable steady states. There are two important differences. First, my mechanism is distinct from theirs: it does not rely on sticky wage or aggregate demand externalities. Second, my model delivers unique predictions: it accounts for the post-recession pattern of the labor wedge. [35] successfully delivers rise in the efficiency wedge but has nothing to say about labor wedge. [34] delivers a sharp rose in the labor wedge, but the rise is entirely due to the "households component," namely, the rise of the wedge between the marginal rate of substitution and the real wage. My model predicts that both the "households component" and the "firm component" of the labor wedge rised persistently after the Great Recession, in line with the data.[8]

From an empirical point of view, the paper is related to the empirical literature that estimates the value of the elasticity of substitution between housing and consumption. The empirical literature has reached little consensus on the magnitude of this parameter. On the one hand, studies based on macro-level data frequently find a value greater than unity ([36] and []mms). On the other hand, studies based on micro-level data typically find a value between 0.1 and 0.6 ([38], [16], [39], and []ll). In the quantitative section of this paper, I calibrate this parameter to the middle of the micro studies. I view the results obtained in this paper as an illustration of the importance of this parameter in the macro-finance literature, and therefore urge more research in this area to identify a more precise range for this parameter.

2.2 Theory

This section describes a stylized model that illustrates the mechanism in the simplest possible environment. Time is discrete and runs to infinity. The commodity space consists of a numeraire

[7] Specifically, [33] assumes that every period households save an *exogenous* fraction of their total income.

[8] The households component is the difference bewteen the labor wedge and its firm component. [20] argues that historically, most of the variations in labor wedge come from the households component. This is not true in the recent recession, where there is also a large and persistent spike in the firm component.

consumption good, physical capital, labor, and "land", where land is a good in fixed supply, can be enjoyed by households, and serves as collateral for the firm. The economy is closed, in the sense that all prices, including the interest rate, are endogenously determined.

The economy is populated by a continuum of infinitely lived, identical household-entrepreneurs. In every period, they choose how much to consume and how much to invest in physical capital, land, and bonds. They also choose, as households, how much labor to supply to the market and, as entrepreneurs, how much labor to demand from the market. The flow budget constraint is given by:

$$c_t + p_t h_t + b_t + k_t \leq w_t l_t + \pi_t + p_t h_{t-1} + q_t b_{t+1} + (1-\delta) k_{t-1} \tag{2.1}$$

where c_t denotes consumption, h_t denotes land holdings, p_t denotes the price of land, b_t denotes bonds and q_t denotes the price of bond, k_t denotes the level of capital, l_t denotes labor *supplied* and w_t denotes the wage level. π_t denotes profit earned by the firm:

$$\pi_t = \max_{l_t^d} A(l_t^d)^{1-\alpha} k_{t-1}^{\alpha} - w_t l_t^d \tag{2.2}$$

Similar to [16] and [15], I assume that there is a cash flow mismatch, such that part of the wage bills must be paid in advance of production. Thus, households need to borrow *intraperiod* loans to finance the wage bill. The total amount of borrowing, including interperiod loans and intraperiod loans, cannot exceed some fraction of the market value of the collateral owned by the households, which consists of housing and capital. This gives us the following borrowing constraint:

$$q_t b_{t+1} + \theta w_t l_t^d \leq \xi p_t h_t + \kappa k_t \tag{2.3}$$

Compared to [16] and [15], the novel ingredient here is that land can be used as collateral to finance firm borrowing (including working capital).

Formally, the households' problem can be written as following:

$$\max_{c,h,l,l^d,k} \sum_{t=1}^{\infty} \beta^t U(c_t, l_t, h_t) \tag{HH}$$

Subject to:

$$\begin{aligned}
c_t + p_t h_t + b_t + k_t &\leq w_t l_t + \pi_t + p_t h_{t-1} + q_t b_{t+1} + (1-\delta) k_{t-1} && (2.4)\\
\pi_t &= \max_{l_t^d} A(l_t^d)^{1-\alpha} k_{t-1}^{\alpha} - w_t l_t^d && (2.5)\\
q_t b_{t+1} + \theta w_t l_t^d &\leq \xi p_t h_t + \kappa k_t && (2.6)\\
0 \leq l_t &\leq l_0, c_t, h_t, k_t \geq 0, h_0, k_0 \text{ given}
\end{aligned}$$

The preference is a variation of the standard GHH preference that incorporates taste for housing:

$$U(c,l,h) = \frac{\left(c - \chi \frac{l^{1+1/\nu}}{1+1/\nu}\right)^{1-\sigma}}{1-\sigma} + \omega \frac{h^{1-\sigma}}{1-\sigma} \tag{2.7}$$

Two comments are in order. First, as a standard GHH preference, there is no wealth effect.[9] Second, the key parameter, σ, is both the intertemporal elasticity of substitution and the (inverse of) intratemporal elasticity of substitution between housing and nonhousing consumption. The intertemporal substitution does not matter at all for the result. Intratemporal elasticity is the crucial parameter that determines the strength of the mechanism.[10]

2.2.1 Market Clearing and Equilibrium

In equilibrium, the markets for goods, labor, bonds, and land are all clear. The goods market clearing condition is:

$$c_t + k_t = Ak_{t-1}^{\alpha} l_t^{1-\alpha} + (1-\delta)k_{t-1}$$

Labor market clearing implies that households' supply is equal to firm demand of labor

$$l_t = l_t^d$$

As the aggregate supply of bonds is zero and all agents are homogeneous, bond market clearing implies:

$$b_t = 0$$

This is different from typical models with financial frictions that focus on *intertemporal* borrowing. Here, intertemporal borrowing is always zero in equilibrium. The model therefore highlights

[9] One could assume an alternative GHH formulation, as in [41]:

$$\frac{\left[\left[c^{1-1/\eta} + \omega h^{1-1/\eta}\right]^{1/(1-1/\eta)} - \chi \frac{l^{1+1/\nu}}{1+1/\nu}\right]^{1-\sigma}}{1-\sigma}$$

Unfortunately, this formulation does not completely get rid of the wealth effect as consumption still enters into the households labor supply first order condition.

[10] Alternatively, one could consider a preference that seperates the intertemporal and intratemporal substitution by assuming a CES form with respect to composite consumption and housing:

$$\frac{\left[\left[(c - \chi \frac{l^{1+1/\nu}}{1+1/\nu})^{1-1/\eta} + \omega h^{1-1/\eta}\right]^{1/(1-1/\eta)}\right]^{1-\sigma}}{1-\sigma}$$

The multiplicity result still holds in this environment. Thus, without loss of generality, I consider the simple case where $\eta = 1/\sigma$ and this formulation collapses to 2.7.

the role *intraperiod* borrowing plays. One can extend the framework in various ways so that there is equilibrium intertemporal borrowing, but at the cost of model complexity.

Finally, land is in fixed supply h_0, and therefore land market clearing implies:

$$h_t = h_0$$

By the Warlas Law, we only require that labor, bond, and land markets clear and that goods market clearing follows.

The definition of equilibrium is standard: Agents optimize and markets clear.

Definition 2.2.1 *A competitive equilibrium is a sequence of allocations* $\{c_t, k_{t+1}, h_{t+1}, l_t, l_t^d, b_t\}_{t=1}^{\infty}$ *and a sequence of prices* $\{p_t, w_t, q_t\}_{t=1}^{\infty}$ *such that:*

1. *Given prices, allocations solve the households problem HH.*
2. *Housing, bond, and labor markets clear every period:* $h = h_0, b = 0, l = l^d$

A *steady state* is a competitive equilibrium where the capital stock k_t is time-invariant.

2.2.2 Steady-State Multiplicity

The purpose of this section is to formally establish that there exist multiple locally stable steady states under appropriate assumptions.

Theorem 2 *Suppose* σ *and* ν *are sufficiently big. Then there exists an open set* $U \in R^2$ *such that for any combinations of loan to value ratios* $(\kappa, \xi) \in U$*, there exists more than one locally-stable steady states.*

Proof. See appendix. ■

The theorem says that if the intratemporal elasticity of substitution is sufficiently low (big σ) and the Frisch elastity of substitution is sufficiently high (big ν), then multiple steady states arise. The formal proof takes two steps. First, I need to prove that there exists multiple steady states and second prove that these steady states are locally stable. The formal proof is delegated to the appendix. here I present a heuristic proof.

There are five steady state equations characterizing the steady state variables: capital, consumption, wage, labor, and land price (k, c, w, l, p).

$$\beta \left[A\alpha k^{\alpha-1} (l)^{1-\alpha} + (1-\delta) + \left(\frac{(1-\alpha)Ak^{\alpha} (l)^{-\alpha}}{w} - 1 \right) \kappa \right] = 1 \qquad \text{(Capital FOC)}$$

The first equation, Capital FOC, is the intertemporal first-order condition of capital. Compared to the first-order condition of capital arising from a standard neoclassical model, there is a novel term, $\left(\frac{(1-\alpha)Ak^{\alpha}(l)^{-\alpha}}{w}-1\right)$. This term captures the fact that accumulating capital helps to relax borrowing constraints whenever they are binding. To see this, note that the numerator, $(1-\alpha)Ak^{\alpha}\left(l\right)^{-\alpha}$, is the marginal product of labor, and the denominator is the real wage. When the borrowing constraint is slack, the marginal product of labor is equal to the wage, and this term vanishes. This term only shows up when borrowing constraints are binding, and thus there is a positive wedge between the marginal product of labor and the wage.

$$l^{\frac{1}{\nu}} = w \qquad \text{(Labor Supply)}$$

$$\min\left(\frac{\xi p h_0 + \kappa k}{w}, \left(\frac{(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}} k\right) = l \qquad \text{(Labor Demand)}$$

Equation Labor Supply and Labor Demand characterize the labor supply decision of the households and the labor demand decision of the firm respectively. Note that the wealth effect is absent in equation Labor Supply : households labor supply is only determined by wage and is independent of households' consumption. In equation Labor Demand, firm's labor demand is potentially constrained by the total value of assets it owns. Thus, its labor demand is given by the minimum of its employment capacity determined by collateral value $\frac{\xi p h_0+\kappa k}{w}$, and its unconstrained labor demand $\left(\frac{(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}} k$.

And there is an aggregate Resources Constraint:

$$c + \delta k = Ak^{\alpha} l^{1-\alpha} \qquad \text{(Resources Constraint)}$$

Finally, there is an intertemporal first-order condition for land:

$$\omega h_0^{-\sigma} + \left(\beta p + \left(\frac{(1-\alpha)Ak^{\alpha}\left(l\right)^{-\alpha}}{w} - 1\right)\xi p\right)\left(c - \chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-\sigma} = p\left(c - \chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-\sigma} \qquad \text{(Land FOC)}$$

The left-hand side is the marginal benefit of purchasing additional units of land. Its consists three components. First, there is intrinsic benefit from owning land, as shown in the first term. Second, one can resell the land tomorrow and get its resell price βp. Third, land could serve as collateral and relax the borrowing constraints of the firm, as shown in the third term. The right-hand side is the marginal cost, which is equal to the current price of land. The first-order condition equates the marginal benefit of land to the marginal cost.

We proceed to characterize the system. To do so, we first observe that Equations Capital FOC, Labor Supply, Labor Demand, and Resources Constraint define an implicit mapping from land price p to other equilibrium objects k, w, l, and c. This mapping generally does not have a closed-form but we can evaluate its derivatives through implicit differentiation. Write this mapping heuristically as $k(p), w(p), l(p), c(p)$. Next, we define the following function F which is the left hand side of Land FOC less its right hand side, divided by the shadow value of wealth $\left(c - \chi \frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-\sigma}$:

$$F(c, l, w, k, p) := \omega h_0^{-\sigma} \left(c - \chi \frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma} + \beta p + \left(\frac{(1-\alpha)Ak^{\alpha}\,(l)^{-\alpha}}{w} - 1\right)\xi p - p \tag{2.8}$$

Plug the mapping $k(p), w(p), l(p), c(p)$ into function F, we arrive at one equation with one unknown:

$$f(p) := F(c(p), l(p), w(p), k(p), p) \quad = \quad \underbrace{\omega h_0^{-\sigma} \left(c(p) - \chi \frac{l(p)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma}}_{\text{Intrinsic benefit from land}} + \underbrace{\left(\frac{(1-\alpha)Ak(p)^{\alpha} l(p)^{-\alpha}}{w(p)} - 1\right)\xi p}_{\text{Relaxing credit constraint}} - \underbrace{(1-\beta)p}_{\text{Net cost}} = 0$$

I summarize the above observations into the following lemma:

Lemma 2.2.1 *Equations Capital FOC, Labor Supply, Labor Demand, Resources Constraint define a mapping* $p \to (k, l, w, c)$. $\{p, k(p), w(p), l(p), c(p)\}$ *is a steady state if and only if* $f(p) = 0$

The function f has a natural interpretation. It measures the households' *excess* willingness to pay for an additional unit of land in addition to its net cost. The excess-willingness-to-pay function f has three components. The first component captures the intrinsic benefit from owning the land. The second component captures the fact that owning more land helps to relax the working capital constraint. The last component is the net cost of land, which takes into account that one can sell the land at price p tomorrow. All components are measured in current consumption units as the original land first-order condition is divided by the shadow value of wealth.

In Figure 2.3, I plot a typical function $f(p)$ with certain parameter values. Due to the nonmonotonicity of f there are multiple crossings with zero axes. Hence, by lemma 2.2.1, there exist multiple roots to equation $f(p) = 0$.

What makes function f nonmonotoic with respect to land price p? Note that in a frictionless environment this cannot arise. Function f would always be decreasing with respect to the land price p due to the usual price effect: Households are less willing to pay for land as the net cost of land increases.

Here, in the presence of the working capital constraint this monotonicity property no longer holds. In particular, there exists a region where households are more willing to buy land when the land gets *more expensive*. This is due to a novel general equilibrium effect through the working capital constraint: When the working capital constraint is binding, an increase in the land price increases households consumption. This reduces the shadow value of wealth and makes households more willing to pay for land. Formally:

Lemma 2.2.2 *(With binding working captail constraint, increases in land price raise consumption)*
The mapping $p \to (k, l, w, c)$ defined by Equations Capital FOC, Labor Supply, Labor Demand, Resources Constraint has the following property:

$$\frac{\partial c(p)}{\partial p} \geq 0$$

The inequality is strict if the working capital constraint is binding:

$$\frac{\xi p h_0 + \kappa k(p)}{w(p)} < \left(\frac{(1-\alpha)A}{w(p)}\right)^{\frac{1}{\alpha}} k(p)$$

With a binding working capital constraint, an increase in land price p raises consumption because it raises output. The output boom arises because firms are allowed to hire more workers with enhanced employment capacity and, with greater incentives to invest, possess higher levels of steady state capital. The next lemma states that increases in consumption raises households' willingness to pay for land, *holding other equilibrium quantities constant.*

Lemma 2.2.3 *(Raising consumption raises households' willingness to pay for land)*

$$\frac{\partial F}{\partial c} > 0$$

Lemma 2.2.3 states that with greater consumption (and also greater composite consumption[11]), the shadow value of wealth, $\left(c(p) - \chi \frac{l(p)^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-\sigma}$, decreases. This increases the value of function f, the households' excess willingness to pay for land. Thus, Lemma 2.2.2 and Lemma 2.2.3 collectively form a complete logical chain of the collateral channel: with binding working capital constraint, increase in land price raises households willingness to pay for land through increased consumption.

Equation 2.9 provides a decomposition of f:

$$\frac{\partial f(p)}{\partial p} = \frac{\partial F(c(p), l(p), w(p), k(p), p)}{\partial p} = \underbrace{\underbrace{\frac{\partial F}{\partial c}}_{>0 \text{ By lemma 2.2.3}} \overbrace{\frac{\partial c}{\partial p}}^{>0 \text{ By lemma 2.2.2}}}_{\text{Collteral Channel}>0} + \underbrace{\frac{\partial F}{\partial p}}_{\text{Usual Price Effect}<0} \tag{2.9}$$

[11] One can show that with a sufficiently large Frisch Elasticity ν, increase in land prices not only increases consumption, but also increases composite consumption.

To summarize, there are two main opposite forces at work. First, there is a conventional negative price effect: When land price becomes more expensive, households are less willing to pay for it. Second, there is a positive collateral effect that is absent from a frictionless model: With binding working capital constraint, increase in land price raises households willingness to pay for land through increased consumption (lemma 2.2.2 and 2.2.3 illustrate the logic chain.). When the collateral effect is strong enough, it overturns the conventional price effect and leads to an upward sloping willingness to pay function f. This is the source of multiple steady states in my model.

2.2.3 Discussion of Assumptions

The strength of this collateral channel depends on the intratemporal elasticity parameter σ and Frisch elasticity of labor ν. The following two propositions illustrate the respective roles of the two parameters.

Proposition 2.2.1 *(Role of Intratemporal Elasticity of substitution σ)*
The function F defined by Equation 2.8 has the following property:

$$\frac{\partial F^2}{\partial c \partial \sigma} > 0$$

The lemma says with bigger σ, changes in consumption c brings about larger changes the households willingness to pay for land through larger changes in the shadow value of wealth, *holding other equilibrium quantities fixed.* This is intuitive. Suppose $\sigma \to 0$, households become risk neutral. Then changes in consumption c do not affect the shadow value of wealth, which is always equal to unity. This breaks down the collateral channel as $\frac{\partial F}{\partial c}$ would equal to zero. Thus, for the collateral channel to work, I need a large σ so that the shadow value of wealth is sensitive to consumption fluctuations.

Proposition 2.2.2 *(Role of Frisch Elasticity of Labor Supply ν)*
The mapping $p \to (k, l, w, c)$ defined by Equations Capital FOC, Labor Supply, Labor Demand, Resources Constraint has the following property:

$$\frac{\partial l(p)^2}{\partial p \partial \nu} > 0, \frac{\partial k(p)^2}{\partial p \partial \nu} > 0, \frac{\partial c(p)^2}{\partial p \partial \nu} > 0$$

With bigger ν, the substitution effect of labor supply is stronger. This implies that changes in land price has bigger impact on labor ($\frac{\partial l(p)^2}{\partial p \partial \nu} > 0$), on capital ($\frac{\partial k(p)^2}{\partial p \partial \nu} > 0$), and therefore on consumption ($\frac{\partial c(p)^2}{\partial p \partial \nu} > 0$). This is intuitive as well. Imagine that $\nu \to 0$. Then labor is approximately inelastically supplied. Thus, equilibrium labor is insensitive to land price fluctuations. So is equilibrium consumption. This breaks down the collateral channel as $\frac{\partial c}{\partial p}$ would equal to zero.

The dual role of land is crucial for multiple steady states to arise. The following proposition illustrates this point:

Proposition 2.2.3 *(The dual role of land is important for steady-state multiplicity)*
When either of the following assumptions hold, f is monotonically decreasing and there exists a unique steady state:

1. $\omega \to 0$ *(When the consumption role of land vanishes)*
2. $\xi \to +\infty$ *(When the collateral role of land vanishes)*

When the land taste parameter ω tends to 0, households do not enjoy land as consumption. This implies consumption fluctuations become irrelevant for land price fluctuations. This in turn weakens the collateral channel and leads to a unique steady state. When the loan-to-value ratio ξ tends to infinity, credit constraints no longer bind. The weakened collateral role of land undermines the collateral channel, leading to a unique steady state as well.

Finally, I briefly discuss the technical steps of the proof. To show that there exist multiple steady states, one must show that there exists three points $p_1 < p_2 < p_3$ such that $f(p_1) > 0, f(p_2) < 0, f(p_3) > 0$. This, coupled with the continuity of $f(p)$, implies that there are multiple steady states. To show that there exists such p_1, p_2, p_3, we need to take the following steps (figure B.3):

1. Solve the unique frictionless steady state (steady state in an economy without working capital constraints). Denote the steady state $(c_{ss}, k_{ss}, p_{ss}, w_{ss}, l_{ss})$.
2. If κ is sufficiently small, define $\xi_{ss} = \frac{w_{ss}l_{ss} - \kappa k_{ss}}{p_{ss}h_0}$. Show that given ξ_{ss}, there exists multiple nontrivial steady states: (See the red curve of figure B.3)
   (a) $f(0; \xi_{ss}) > 0$
   (b) $f(p_{ss}; \xi_{ss}) = 0$ and $f'^{-}(p_{ss})$, where f'^{-} denotes the left derivative of f ;
3. Show that the case extends to $\xi \in (\xi_{ss}, \xi_0]$ for some ξ_0.

Having established that there exists multiple steady states, we next proceed to examine their local stability. A formal proof is given in the appendix. Here we heuristically argue that, the biggest and the smallest steady states in figure 2.3 are locally stable whereas the middle one is unstable. To see why, focus on the smallest steady state, call it p_0. Perturb it downwards to $p_0 - \delta$. One can see that $f(p_0 - \delta) > 0$ as f is downward sloping locally around p_0. Thus households are willing to pay more for land, pushing the land price upward. Similarly if the land price is perturbed upwards, households willingness to pay would drop, pushing the land price back to p_0. Thus, p_0 is locally stable. The logic holds for the biggest steady state and is reversed for the middle steady state. Of course, to prove local stability, one need to write donw the dynamic system and examine the eigenvalues of the transition matrix, similar to []slp. We delegate all the formal proofs to the appendix.

2.3 Extended Model and Quantitative Analysis

The model in section 2.2 is stylized, and serves the purpose of illustrating the mechanism and highlighting the required assumptions. To summarize, I demonstrated that the steady-state multiplicity arises if 1) the housing and nonhousing consumption are not very substitutable; and 2) the substitution effect on labor supply is strong.

To keep the theoretical result clean and transparent, the stylized model imposes some restrictions on model structure. For instance, there is no distinction between residential land and commercial land. All land is owned by the representative households-entrepreneur. Moreover, the housing rental market is assumed away, which is a nontrivial proportion of the housing market. Finally, land does not have a productive role.

In view of this, I take on two tasks in this section. First, I propose an extended model in which 1) there is a distinction between residential and commercial land; 2) there is a housing rental market; and 3) land can serves as a factor of production, as in [27] and [26]. Despite richer ingredients, the extended model behaves "very similar" to the stylized model in the sense that an equivalence result can be proved: that the equilibrium allocations and prices in the extended model are the same as those in the stylized model under empirically plausible parameter restrictions. Second, I conduct a quantitative analysis whereby I calibrate the extended model to the US economy and ask the following questions: Does the steady-state multiplicity result still survive? How much persistence can the model generate quantitatively? In particular, is the model able to account for the slow recovery after the Great Recession?

2.3.1 An Extended Model

The economy is populated by two types of agents: a continuum of households and a continnum of firms. The households maximize their lifetime utility $\sum_{t=1}^{\infty} \beta^t U(c_t, l_t, h_t)$, where c_t is consumption, l_t is labor, h_t is the amount of land *enjoyed* by the households, which consists of land owned by them h_{ht} and land rented by them h^r_{ht}. Households are the owners (shareholders) of the firms. The household's budget constraint is:

$$c_t + p_t h_{ht} + r_t h^r_{ht-1} \leq p_t l_{ht-1} + d_t + w_t l_t \tag{2.10}$$

where r_t is the rental rate of land and d_t is the dividend paid out by the firm.

The households problem is to maximize their lifetime utility subject to their budget constraint Equation 2.10 and given the initial land holding h_{h0}:

$$\max \quad \sum_{t=1}^{\infty} \beta^t U(c_t, l_t, h_t)$$

$$c_t + p_t h_{ht} + r_t h^r_{ht-1} \leq p_t l_{ht-1} + d_t + w_t l_t$$

$$h_{h0} \text{ given}, l_t \leq \bar{l}$$

I use the following utility function:

$$U\left(c,l,h\right)=\frac{\left[\left[\omega(c-\chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}})^{1-1/\sigma}+\left(1-\omega\right)h^{1-1/\sigma}\right]^{1/(1-1/\sigma)}\right]^{1-\eta}}{1-\eta} \tag{2.11}$$

Note that this utility function is just an extension of Equation 2.7 where I separate the intertemporal and intratemporal elasticity parameters. Thus I need to introduce an additional parameter η into the model. Note that when $\eta = 1/\sigma$, Equation 2.11 collapses to Equation 2.7.

The firms maximize the discounted sum of dividends weighted by some pricing kernel, M_t, which in equilibrium is equal to the household's intertemporal marginal rate of substitution. Their gross revenue function is given by $F(k,l,h) = Ak^{1-\alpha-\gamma}l^{\gamma}h^{\alpha}$. Note that land has a productive role as in [27] and [26]. The elasticity of output with respect to land is captured by parameter α.

The firms accumulate capital and land. Capital depreciates at rate δ. Land does not depreciate and can be rented out to households at rate r_t. Thus the firms face a tradeoff between land allocated to productive use and rental use. Their budget constraints are given by:

$$d_t + k_t + p_t h_{ft} \leq F(k_{t-1}, l_t, h^p_{ft-1}) + p_t h_{ft-1} - w_t l_t + r_t h^r_{ft} + (1-\delta)k_{t-1}$$

$$h_{ft} = h^p_{ft} + h^r_{ft}$$

where h_{ft-1} is firms' total land holding in the beginning of period t. They can allocate it into two alternative uses: production h^p_{ft-1} and rental h^r_{ft-1}. As in the stylized model, we assume that there is a working capital constraint arising from the cash-flow mismatch problem of the firms:

$$w_t l_t \leq \xi(p_t h^f_t + k_t) \tag{2.12}$$

The firms maximize weighted discounted dividend subject to budget constraints and working capital constraints, given an initial holding of captial and land:

$$\max \quad \sum_{t=1}^{\infty} M_t d_t$$

$$d_t + k_t + p_t h_{ft} \leq F(k_{t-1}, l_t, h^p_{ft-1}) + p_t h_{ft-1} - w_t l_t + r_t h^r_{ft} + (1-\delta)k_{t-1}$$

$$h_{ft} = h^p_{ft} + h^r_{ft}$$

$$w_t l_t \leq \xi(p_t h_{ft} + k_t)$$

$$h_{f0}, k_0 \text{ given}$$

Note that I impose no restrictions on the behavior of households and firms, in particular how they accumulate and use land. How much land is owned by the households and by the firms,how much land is allocated to rental relative to production are all determined in equilibrium.

The definition of competitive equilibrium is standard:

Definition 2.3.1 *A competitive equilibrium is* $\{c_t, l_t, h_{ht}, h^r_{ht}, k_t, h_{ft}, h^r_{ft}, d_t, h^d_t\}_{t=1}^{\infty}$ *and* $\{p_t, w_t, r_t\}_{t=1}^{\infty}$ *such that:*

1. *Given prices, allocations solve the households problem.*
2. *Land, labor, and land rental market clears every period:* $h_h + h_f = h_0, l = l^d, h^r_h = h^r_f$

2.3.2 An Equivalence Result

In this section, we will prove the following equivalence result:

Theorem 3 *Suppose the land's share in production* $\alpha = 0$*. Then the equilibrium allocations (consumption, labor, capital, and investment) and prices in the extended model are the same as those in the stylized model.*

The theorem claims that when the productive role of land is assumed away ($\alpha = 0$), the extended model is equivalent to the stylized model in terms of major equilibrium allocations and prices. What, then, is the empirically plausibe range for α? The general consensus of the literature is that it is not too much different from zero. [27] and [26], for instance, set this parameter to 0.03. This, together with the equivalence result, suggests that the dynamics in the extended model is not too much different from that in the stylized model, and in particular, the steady-state multiplicity result is likely to survive in the extended model.

The proof amounts to compare the set of first order conditions and resources constraints across the two models. Before we proceed, the following lemma greatly simplifies the characterization of equilibrium in the extended model:

Lemma 2.3.1 *There exists an equilibrium in the extended model such that all land is owned by the firms. Equilibrium allocations (consumption, labor, capital, and investment) and prices in other equilibria, if any, are the same as those in this equilibrium. If the working capital constraints bind at any date, this is also the unique equilibrium.*

This lemma suggests that it suffices to restrict attention to the equilibrium where all land is owned by the firm. The intuition is as follows. In the presence of working capital constraints, it is always *weakly* socially optimal to allocate land to the firms, as this always weakly relaxes the credit constraints. Specifically, when the working capital constraints do not bind at any future dates, there exists a continuum of equilibria indexed by the distribution of land between the households and the firms. In each of these equilibria, households are indifferent between owning or renting a house. These equilibria are equivalent in terms of real allocations. When the working capital constraints do bind at some future dates, firms are willing to pay additional money for land, reflecting the additional benefit of relaxing the (future) credit constraints. This drives households out of the land purchase market and into the rental market. As a result, the unique equilibrium outcome is that all land is owned by the firm.

We take the first order condition for the households with respect to h_h^r:

$$U_c r = U_h \tag{2.13}$$

where U_c is the partial derivative of utility with respect to consumption and U_h is the partial derivative of utility with respect to land. This equation says that the benefit (right hand side) of renting land and enjoy the service flow generated from the land is equal to the rental cost (left hand side). We can also take the first oder condition for the firms with respect to h_f:

$$M'p' + r'M' = Mp \tag{2.14}$$

Where M is the pricing kernal for the firm. And a prime denotes tomorrow's variable. This equation says that the current cost of buying land, in equilibrium, should be equal to the future benefit, including the capital gain of land (first term of left hand side) and the rent (second term of left hand side). Note that the pricing kernal

$$M = \beta^t U_c \tag{2.15}$$

Plug Equation 2.15 and 2.13 into Equation 2.14, one arrive at the following equation:

$$\beta U_c' p' + \beta U_h = U_c p$$

This is exactly the housing first order condition one would obtain from the stylized model. One can also check other first order conditions, resources constraints, and credit constraints and conclude that theorem 3 holds.

2.3.3 Calibration and Computation

In this section I describe the calibration procedure. Most parameters are set to standard values used in the literature or calibrated using standard targets. Aggregate land supply h_0 is normalized to 1, whereas the relative quantity of land owned by the firm $\bar{h}^f$ is set to .5, consistent with [43]. On the household side, the discounted factor β is set to 0.99 as I calibrate the model to quarterly frequency. The intertemporal elasticity of substitution η is set to 2. The Frisch labor elasticity is set to 1, which is in the middle range of micro and macro estimates. The taste parameter ω is calibrated so that the steady state land price to annual GDP ratio is 1.8. The distutility of labor parameter χ is calibrated so that steady state labor is equal to one third. On the production side, productivity A is normalized to 1. Capital share α is set to 0.33 and the labor share γ is set to 0.64. This implies that the land share in the production function is 0.03, value estimated by []lwz and [26]. The depreciation rate δ is equal to 0.025.

We are left with three crucial parameters: the intratemporal elasticity of substitution between housing and nonhousing consumption η, the loan-to-value ratio parameter ξ, and a wage adjustment procedure in order to have realistic fluctuations in employment.

The literature reaches little consensus on what σ should be. On the one hand, studies based on macro-level data frequently find a value greater than unity ([36] and []mms). On the other hand, studies based on micro-level data typically find a value between 0.1 and 0.6 ([38], [16], [39], and []ll). In this section, I set it to 0.33, in the middle of the micro studies.

Next I need to calibrate the loan-to-value ratio parameter ξ. The standard way of calibrating occasionally binding credit constraints is to target the frequency of crises, as in [29]. Here, due to the computation burden, I abstract away from stochastic shocks. Therefore the standard strategy does not apply here. I develop a strategy analogous to the standard one, where ξ is calibrated to meet two criterion. First, the working capital constraint does not bind for "mild" recessions, that is, recessions where output fluctuations are less than 5%, average of postwar recessions except the Great Recession. Second, the working capital constraint binds for the Great Recession where outputs drop by 10%. I pick ξ such that the credit constraint just binds with 8% drop in output, middle of the two criterion. The resulting value for ξ is 0.04.

Finally, I need to incorporate some wage stickiness in order to generate realistic fluctuations in employment. I assume that the real wage is downward rigid:

$$w_t \geq \zeta w_{t-1} \tag{2.16}$$

This real wage adjustment constraint can arise, for example, in an environment with nominal wage rigidity and in which the central bank is reluctant to raise inflation, as in [30]. Here we do not model the details of a monetary economy, but take the constraint as given. The wage adjustment parameter ζ is set to $(\frac{1}{1+2\%})^{1/4} \approx 0.995$. This captures the idea that households are unwilling to accept nominal wage cuts and the central bank is maintaining a 2% inflation. To see why, let W_t denote the nominal wage level and P_t denote the nominal price level. Then real wage is just the ratio of nominal wage and nominal price. As the household is unwilling to accept nominal wage cuts:

$$W_t \geq W_{t-1}$$

And the central bank is maintaining 2% inflation annually:

$$P_t = (1 + 2\%)P_{t-1}$$

dividing the first equation by the second, one has

$$w_t \geq \frac{1}{1 + 2\%} w_{t-1}$$

Therefore annual adjustment is $\frac{1}{1+2\%}$, implying that quarter adjusment is $(\frac{1}{1+2\%})^{1/4} \approx 0.995$.

Note that calibrating ζ with 2% annual inflation serves as a conservative benchmark given that inflation has been low in recent years. If inflation is lower, real wage would display greater stickiness, strengthening our mechanism. []mary documents that wage growth has been strong since the recession started, exactly because inflation has been low. Thus our quantitative results can be thought of as a lower bound on the strength of our mechanism.

Despite no exogenous shocks, computing the recursive competitive equilibrium turns out to be a nontrivial task, as there are two occasionally binding constraints: equation 2.12 and 2.16. Moreover, there are two state variables: capital stock and previous-period wage. There are strong nonlinearities with respect to each state variable, especially with respect to the previous wage. I implement a version of the policy function iteration modified to account for occasionally binding constraints. I also consider uneven-spaced grids, placing more grid points around the region where nonlinearities are more likely to occur. A detailed description of computation algorithm is given in the Appendix. The resulting Eulor Equation error is on the order of 10^{-9}.

2.3.4 Quantitative Results

Figure 2.4 presents the main quantitative result, that there is asymmetry in recovery speed upon small and large shocks. The economy recovers immediately after small shocks, but experiences significant delays upon large shocks. This is in stark contrast to a neoclassical model. For comparison, I also solve a standard neoclassical model without frictions and plot its impulse response to shocks of different sizes. For the standard neoclassical model, capital recovers immediately for both small and big shocks.

To understand the mechanism, figure B.5 plots various equilibrium functions. The crucial observation is that the law of motion for capital is *S-shaped* conditional on previous period wages (the top left panel). As a result, the economy displays asymmetric responses to small and large shocks. In particular, recovery is delayed upon large shocks. In figure 2.5, I provide a heuristic description of why recovery is delayed upon large shocks. The response of the economy to small shocks is identical to a neoclassical model. Suppose the economy operates at the steady state A. In period 0 there is a mild negative shock such that the capital stock in this economy drops to point B. In this region land price drops are mild and therefore credit constraints do not bind.

As a result, the economy immediately recovers—just as a standard neoclassical model would predict.

When there is a large negative shock at period 0, however, the dynamics are different. Suppose that a very large negative shock hit the economy in period 0 and the capital stock drops to point B in period 1. Households wealth drops sharply. Land price falls sharply as well, both today and in the future. But wage cannot fall too much due to the downward wage rigidity constraint. This implies sharp drops in labor, both today and in the future. This, in turn, depresses investment and thus in period 2, the economy's capital stock drops further to point B_1. In period 2, wage starts falling but is still not sufficient whereas at the same time, land price is still very depressed. So investment is still low, dragging the economy further down to point C_2 in period 3. Not until period 4 does wage drop to a sufficiently low level such that employment starts to recover. Thus, with large adverse shocks, the economy experiences delayed recovery: recovery comes with a few period's lag.

Why is the model successful in generating significant persistence after large recessions? It is important to understand the behavior of equilibrium prices. The bottom left panel of Figure B.5 plots equilibrium land price, which is very sensitive to fluctuations in capital. If capital is 80% of its unconstrained steady-state level, land price is only 50% of that. On the other hand, as shown in the bottom right panel of Figure B.5, equilibrium wage is insensitive to capital fluctuations, due to the downward wage rigidity constraint. The highly sensitive equilibrium land price function together with the highly insensitive wage function implies that firm's borrowing constraint has a quantitatively important impact on the dynamics of labor. As one can see in the top right panel of Figure B.5, labor drops 30% when capital falls by about 20%.

It is important to understand the separate roles of sticky wage and land price dynamics in shaping aggregate dynamics. To make the point, I compute an alternative model in which land prices in the credit constraint is exogenously fixed at $p = \bar{p}$. More precisely, the credit constraint is given by:

$$\theta wl \leq \xi(k^{'} + \bar{p}h^{'})$$

In this economy, land price dynamics do not affect agents' borrowing capacity. I label it "constant-p" economy. I set $\bar{p}$ to the unconstrained steady state level p_{ss}. Figure 2.7 plots the policy functions across three economies: the neoclassical economy (black curve), the constant-p economy (blue curve), and the benchmark economy (red curve). I plot four panels corresponding to different level of previous-period wage. As one can see, both the constant-p economy and the benchmark economy exhibit nonlinearities in the policy functions. Thus, both economies are capable of generating more persistence than the neoclassical economy, upon sufficiently large negative shocks. The nonlinearity, however, is much stronger in the benchmark economy where land prices fluctuations affects borrowing constraints. Interestingly, when wages are relatively high (top left panel), there is not much difference between the benchmark economy and the constant-p economy. The difference gets much more substantial when wages are relatively low. For example, when wage is 5 percent below its steady-state level (bottom right panel), capital starts to recover in the constant-p economy but still delines in the benchmark economy.

In figure 2.4, I compare the impulse responses upon small and big shocks, in a frictionless economy, constant-p economy, and the benchmark economy. The right panel plot the impulse responses of small shocks (such that capital only falls by 1 percent). The transition path is almost identical across three economies. The right panel plot impulse responses of big shocks. And one can see delayed recovery both in the constant-p economy and in the benchmark economy. The delays in the "constant-p" economy are mainly due to the sticky-wage constraint, as in [34]. Delays in the benchmark economy are much more significant, and the additional persistence arises because of endogenously tightened borrowing constraints.

2.3.5 Accounting for the aftermath of the Great Recession

Now I conduct a simulation of the Great Recession, to evaluate whether the model is able to generate the slow recovery of major aggregate variables, similar to the data. To do so, I first calibrate the shocks that I later feed into the model. I consider two sources of shocks. The first is financial shocks to the loan-to-value ratio ξ. The second is productivity shocks to A. I use the productivity time series constructed by [19] to calibrate productivity shocks, and calibrate the financial shock ξ such that the model matches the overall drop in housing prices after 2007, which is about 35%. The resulting ξ is ploted in Figure B.4.

I feed the two shocks into the model, and Figure 2.8 demonstrates the behavior of various variables during the recovery phase, which is between 2010 and 2016. The model is broadly consistent with the behavior of various macro variables in the recovery phase. It does a particularly good job of matching the behavior of labor, and it even over-predicts the *slow* recovery of investment compared to the data. Yet, the model under-predicts the recovery speed for output. This is possibly due to the fact that we use a utilization-adjusted TFP series. If we instead use an unajusted TFP time series, the decline in TFP would be more salient, and this could further drag down output.

It would be interesting to see how the credit shock alone contributes to the slow recovery after the Great Recession. Figure 2.9 plots how variables recover when there is only the credit shock, labeled "credit shock only". Relative to the two shocks case, with only credit shock the model predicts quicker recovery of all four variables. This is expected, as productivity has gradually declined since 2008. The model still does a good job of matching the behavior of investment. And it is broadly consistent with the recovery speed for labor, particularly before

2013. after 2013, the model predicts that the recovery speed of labor picks up. This is largely due to the wealth effect on labor supply: when the real wage adjust downward, households get the chance to supply more labor, and they are willing to do so because they are poor. The quantitative performance is expected to be improved if one use preferences without the wealth effect such as the GHH preference. I conclude, that the propagation mechanism described here could play an important role in accounting for the slow recovery after the Great Recession, and particularly the behavior of labor and investment

Accouting for the labor wedge

There has been a sharp and persistent rise in the labor wedge since the Great Recession. The labor wedge is defined as the log distance between the marginal production of labor and the marginal rate of substitution:

$$\text{labor wedge} = log(\text{marginal production of labor}) - log(\text{marginal rate of substitution})$$

. The Great Recession also witnessed a sharp and persistent rise in the *firm component* of the labor wedge, defined as the distance between the marginal product of labor and the real wage:

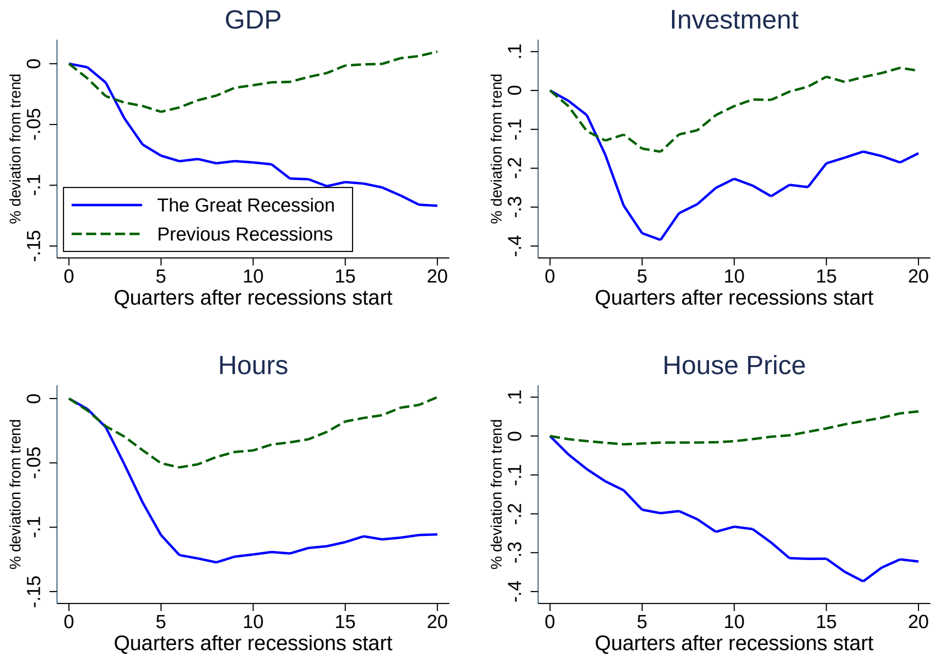
$$\text{Firm component of labor wedge} = log(\text{marginal production of labor}) - log(\text{real wage})$$

As shown in Figure 2.10, my model matches the salient features of the labor wedge and its firm component after the 2008 recession. The rise of the labor wedge in the model is due to the sticky wage constraint, which drives a wedge between the marginal rate of substitution and the real wage (households component), and the credit constraint, which drives a wedge between the real wage and the marginal production of labor (firm component). After the recession, both constraints binds for an extended period of time, leading to persistent rise in the labor wedge and its firm component.

2.4 Conclusion

The Great Recession was very different from other postwar recessions, due not only to big declines during the recession, but also to the slow recovery. This paper aims to advance our understanding of the Great Recession and its aftermath. I propose and quantify a framework in which land serves dual roles: either as household consumption or as firm collateral to finance borrowing, in particular its working capital. Within this framework, the law of motion of capital is S-shaped, leading to existence of multiple steady states and hence asymmetric responses to small and large shocks.

Important future work remains to be done. First, for the sake of clarity, the theory is presented in an intentionally simple framework with representative agents. It would be interesting to explore a fully quantitative framework in which heterogeneous agents are present with different borrowing capacities, calibrated to resemble the US economy. Second, a crucial parameter that determinines the strength of the mechanism is the intratemporal elasticity of substitution between housing and consumption. Unfortunately, so far the literature has reached little consensus on the magnitude of this parameter. Tightening its empirical range is another line of important future work.



Note: This figure plots linearly detrended aggregate variables in the five-year window following the Great Recession (solid blue curve) and previous recessions (dashed green curve). Previous recessions include the 2000 recession, the 1990 recession, the 1981 recession, the 1973 recession, and the 1960 recession. Starting point is normalized to 0. *GDP is the real GDP per capita. Investment is the real private gross investment. Labor is the total hours available from BLS. Housing price is the Case-Shiller real home price index.*

Figure 2.1: The Great Recession Compared to Other Postwar Recessions

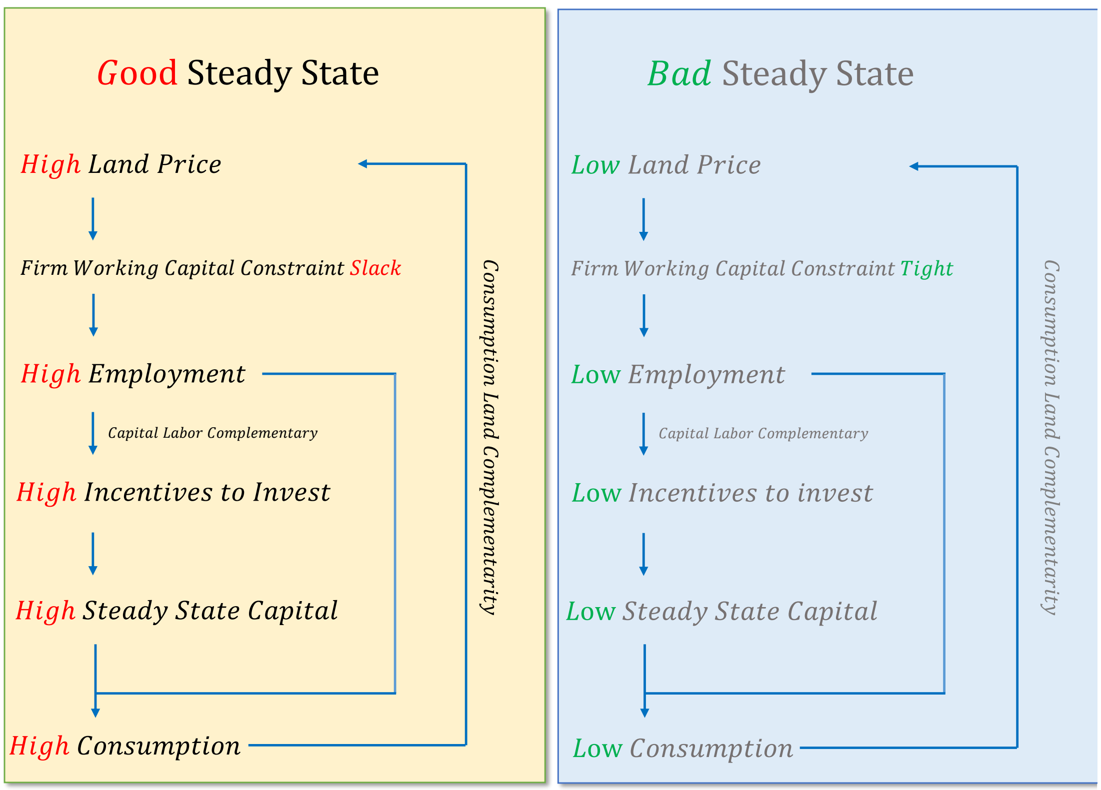
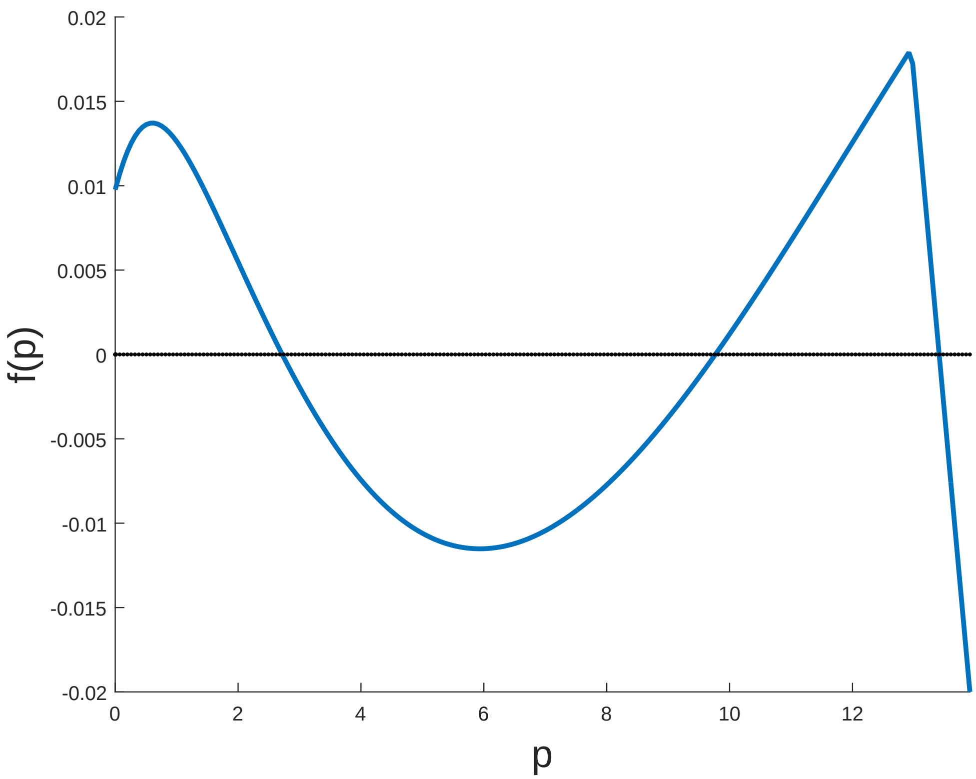
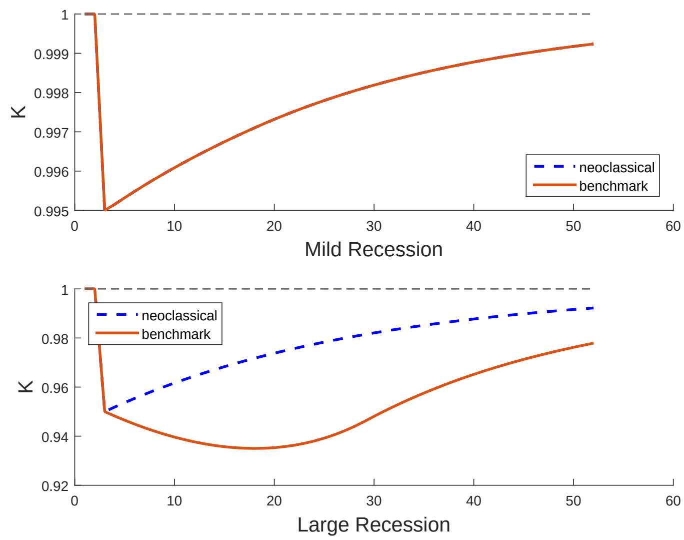


Figure 2.2: Multiple Steady States



This figure plots typical excess-willingness-to-pay-function f. Each crossing with zero axes is a steady state. Thus there are three steady states. The biggest and the smallest ones are both locally stable. Parameters value used: $\beta = 0.96, \delta = 0.1, \sigma = 3, \alpha = 0.35, \omega = 1, \nu = 6, \xi = 0.13, \kappa = 0$

Figure 2.3: Excess-willingness-to-pay-function $f(p)$ and Multiple Steady States



Transitional dynamics starting from .5% *and* 5% *below the steady state level of capital. Model dynamics identical to its neoclassical counterpart with mild loss of capital (top panel), and significant delay in recovery with severe loss of capital (bottom panel).*

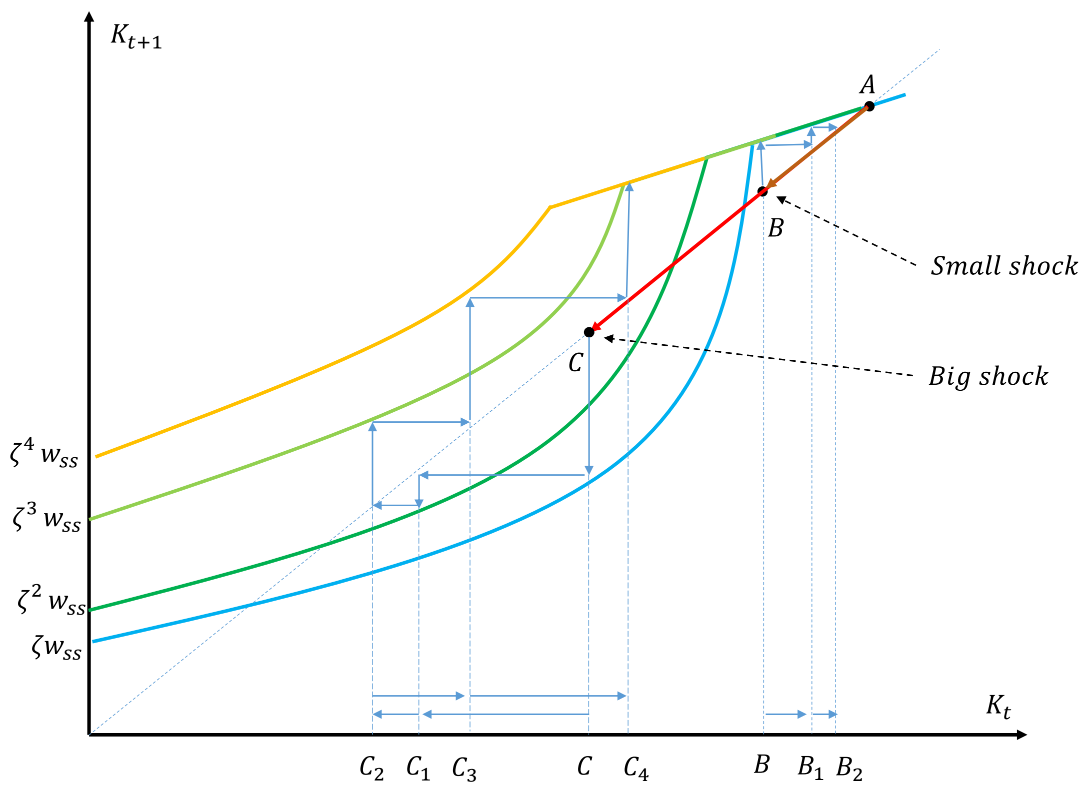
Figure 2.4: Transitional Dynamics with Different Initial Capital

	Output			Investment		
Year	2011	2013	2015	2011	2013	2015
Data	-12.7%	-35.5%	-49.7%	2.9%	26%	35%
Model	-10.7%	-18.8%	-25.6%	-54.4%	-90.0%	-112%

	Labor			Land Price		
Year	2011	2013	2015	2011	2013	2015
Data	4.9%	15.2%	26.9%	-24%	-25%	13.9%
Model	5.9%	17.9%	28.0%	3.4%	10%	18.6%

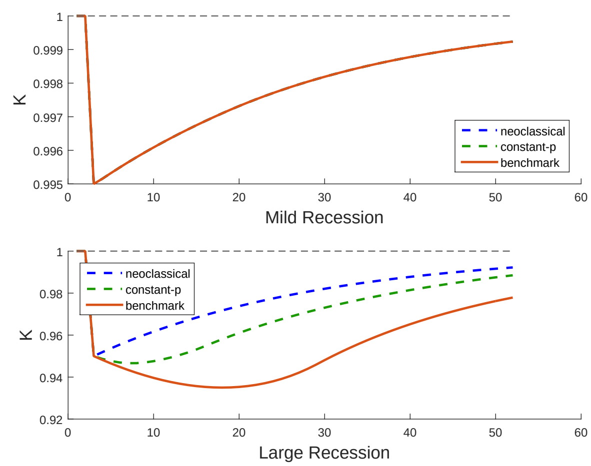
Note: the rate of recovery is defined as the fraction of lost variables recovered relative to the fourth quarter of 2009. For instance, in 2011 4.9% of the lost labor was recovered (bottom left panel). The rate of recovery for output is negative as detrened output kept declining after the recession. The model did a particularly good job in matching the behavior of labor. It also predicts the declining post recession output and the pace of recovery for land price at longer horizon.

Table 2.1: Comparing the Rate of Recovery



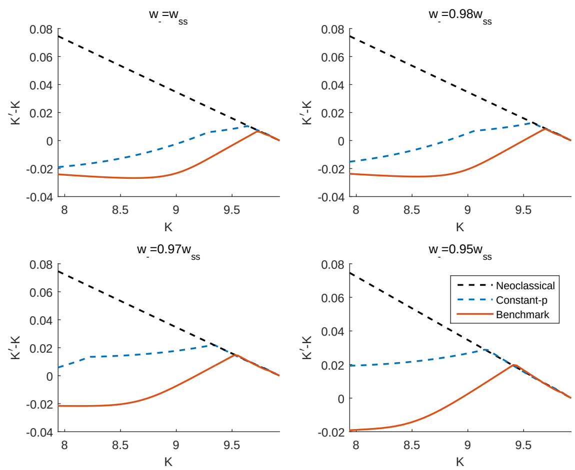
This figure plots typical law of motion for capital and highlights different dynamics upon small and large shocks. Suppose the economy operates at the usual steady state point A. After a mild recession, the level of aggregate capital stock drops to point B. Then it recovers immediately $B \to B_1 \to B_2 \ldots$. In contrast, suppose the economy is hit by a severe negative shock so that its capital stock ends up at point C. Then it will drop further to C_1 and then C_2, finally recovery in period 3. Thus, recovery after big recessions is significantly delayed.

Figure 2.5: Graphical Illustration of Delayed Recovery



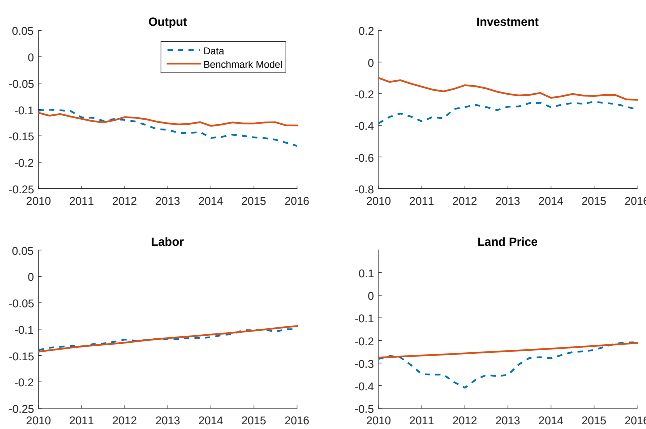
Transitional dynamics starting from .5% *and* 5% *below the steady state level of capital. For the benchmark model, the dynamics is identical to its neoclassical counterpart with mild loss of capital (top panel), and there is significant delay in recovery with severe loss of capital (bottom panel). For the constant-p economy, there is also delays in recovery compared to its neoclassical counterpart, but the delay is less significant.*

Figure 2.6: Transitional Dynamics



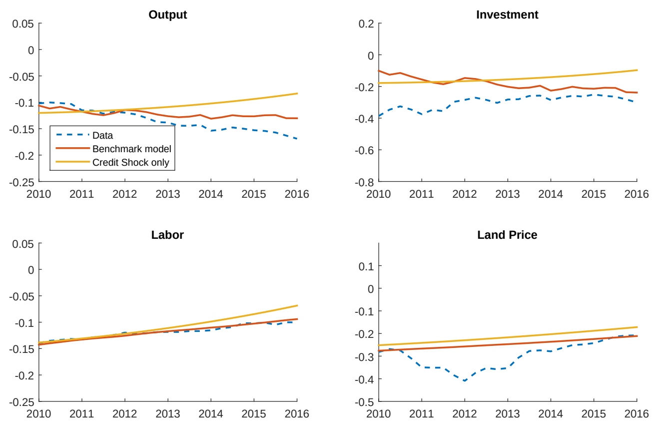
This figure compares policy functions across three models: the standard neoclassical model.Both the constant-p economy and the benchmark economy exhibit nonlinearities in the policy functions. The nonlinearity, however, is much stronger in the benchmark economy compared to the constant-p economy, especially when the previous-period wage is relatively low.

Figure 2.7: Policy Functions



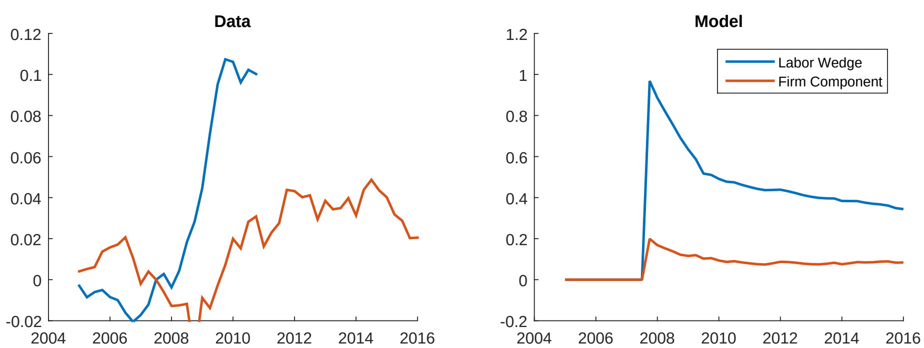
This figure presents simulation results and compares them with the data starting in the year 2010. The data part I use the same time series as in Figure 2.1. For the model part, I feed in the economy with a series of productivity shocks from [19] and a temporary credit shock, such that land price drops by 30%.

Figure 2.8: Simulation I



This figure presents simulation results and compares them with the data. For the data part I use the same time series as in Figure 2.1. For the model part, I feed in the economy with a series of productivity shocks from [19] and a temporary credit shock such that land price drops by 30%. *For the case with credit shock only, I only feed in the credit shocks.*

Figure 2.9: Simulation II



This figure presents simulation results and compares them with the data.

Figure 2.10: Labor Wedge and its Firm Component

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Appendix A

Appendix to Chapter 1

Proof of lemma 1.3.1

I prove existence by construction. Namely, I find a system of equations that fully characterizes exogenous-information steady states $\Phi(\lambda)$ given any value of λ. The method I use to look for such a system of equation is very similar to Wang (1994): look for a system of equations of $\left(\Sigma, p_{\hat{F}}, p_F, p_x\right)$ given λ. Σ is the uninformed's prior of the state variable (F, x) and is pinned down by the Kalman filter equations, given $p_{\hat{F}}, p_F, p_x$. $p_{\hat{F}}, p_F, p_x$ are pinned down by the market clearing condition, given Σ. The detailed proof is as follows.

First, we would like to look for an equation characterizing Σ. The state s_t evolves and the signal is given by the following dynamic system:

$$
\begin{aligned}
s_{t+1} &= As_t + w_{t+1}, \text{ where } w_{t+1}\tilde{}N(0, Q) \\
y_{t+1} &= Gs_{t+1} + v_{t+1}, \text{ where } v_{t+1}\tilde{}N(0, R),
\end{aligned}
$$

where

$$
s_t = \begin{bmatrix} F_t \\ x_t \end{bmatrix}
$$

$$
y_t = \begin{bmatrix} S_{t+1}^P \\ D_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} p_F F_{t+1} - p_x x_{t+1} \\ F_{t+1} + \varepsilon_{t+1}^D \\ F_{t+1} + \varepsilon_{t+1}^S \end{bmatrix}
$$

$$
\begin{aligned}
A &= \begin{bmatrix} \rho^F & 0 \\ 0 & \rho^x \end{bmatrix} \\
G &= \begin{bmatrix} p_F & -p_x \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \\
Q &= \begin{bmatrix} \sigma_F^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix} \\
R &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_D^2 & 0 \\ 0 & 0 & \sigma_S^2 \end{bmatrix}.
\end{aligned}
$$

Apply the formula of the Kalman filter:

$$
\begin{aligned}
\hat{s}_{t+1} &= A\hat{s}_t + \left[A\Sigma_t A'G' + QG'\right]\left[GA\Sigma_t A'G' + GQG' + R\right]^{-1}(y_{t+1} - GA\hat{s}_t) \qquad \text{(A.1)} \\
\Sigma_{t+1} &= A\Sigma_t A' + Q - \left[A\Sigma_t A'G' + QG'\right]\left[GA\Sigma_t A'G' + GQG' + R\right]^{-1}\left[A\Sigma_t A'G' + QG'\right]'.
\end{aligned}
$$

At a stationary equilibrium, Σ must be stationary over time, and therefore the second equation becomes

$$
\Sigma = A\Sigma A' + Q - \left[A\Sigma A'G' + QG'\right]\left[GA\Sigma A'G' + GQG' + R\right]^{-1}\left[A\Sigma A'G' + QG'\right]'. \qquad \text{(A.2)}
$$

We arrive at the first equation, which characterizes the uninformed's conditional expectation Σ.

Next, we know that investors solve the following problem:

$$
\max_e -E\left[e^{-\alpha Rw - \alpha e\left(D'+P'-RP\right)}|\Omega^i\right], i = u, i.
$$

Thus, its demand is given by

$$
\begin{aligned}
D^i &= \frac{E\left[D'+P'|\Omega^i\right] - RP}{\alpha Var\left[D'+P'-RP|\Omega^i\right]} \qquad \text{(A.3)} \\
&= \frac{E\left[D'+P'|\Omega^i\right] - RP}{\alpha Var\left[D'+P'|\Omega^i\right]},
\end{aligned}
$$

where the second equality follows because the current price P is in the information set of both agents. Note that $E\left[D'+P'|\Omega^i\right]$ and $Var\left[D'+P'-RP|\Omega^i\right]$ are the conditional mean and variance of the excess stock return perceived by the agent conditional on its information set Ω^i. We can write out $D'+P'$:

$$
\begin{aligned}
D'+P' &= F' + \varepsilon^{D\prime} + a + p_{\hat{F}}\hat{F}' + p_F F' - p_x x' \qquad \text{(A.4)} \\
&= a + (1+p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon^{D\prime}.
\end{aligned}
$$

Note that by the previous Kalman filter expression,

$$
\begin{aligned}
\hat{F}' &= \rho^F \hat{F} + \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix} \begin{bmatrix} p_F \left(F' - \rho^F \hat{F}\right) - p_x \left(x' - \rho^x \hat{x}\right) \\ F' - \rho^F \hat{F} + \varepsilon^{D\prime} \\ F' - \rho^F \hat{F} + \varepsilon^{S\prime} \end{bmatrix} \qquad \text{(A.5)} \\
&= \rho^F \hat{F} + Y_1 \left[p_F \left(F' - \rho^F \hat{F}\right) - p_x \left(x' - \rho^x \hat{x}\right)\right] + Y_2 \left[F' - \rho^F \hat{F} + \varepsilon^{D\prime}\right] + Y_3 \left[F' - \rho^F \hat{F} + \varepsilon^{S\prime}\right] \\
&= \left(\rho^F - Y_1 p_F \rho^F - (Y_2 + Y_3)\rho^F\right) \hat{F} + (Y_1 p_F + Y_2 + Y_3) F' - Y_1 p_x x' + Y_1 p_x \rho^x \hat{x} + Y_2 \varepsilon^{D\prime} + Y_3 \varepsilon^{S\prime},
\end{aligned}
$$

where $\begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}$ denotes the first column of matrix $[A\Sigma A'G' + QG'][GA\Sigma A'G' + GQG' + R]^{-1}$. To ease exposition, define

$$
\begin{aligned}
e_1 &= p_{\hat{F}} \left(\rho^F - Y_1 p_F \rho^F - (Y_2 + Y_3)\rho^F\right) = p_{\hat{F}} \rho^F (1 - Y_1 p_F - Y_2 - Y_3) && \text{(A.6)} \\
e_2 &= p_{\hat{F}} Y_1 p_x \rho^x && \text{(A.7)} \\
e_3 &= \rho^F \left(1 + p_F + p_{\hat{F}} (Y_1 p_F + Y_2 + Y_3)\right) && \text{(A.8)} \\
e_4 &= \rho^x \left(p_{\hat{F}} Y_1 p_x + p_x\right) && \text{(A.9)} \\
e_5 &= 1 + p_F + p_{\hat{F}} (Y_1 p_F + Y_2 + Y_3) && \text{(A.10)} \\
e_6 &= p_{\hat{F}} Y_1 p_x + p_x && \text{(A.11)} \\
e_7 &= p_{\hat{F}} Y_2 + 1 && \text{(A.12)} \\
e_8 &= p_{\hat{F}} Y_3 && \text{(A.13)} \\
& && \text{(A.14)}
\end{aligned}
$$

Thus,

$$
D' + P' = a + e_1 \hat{F} + e_2 \hat{x} + e_3 F - e_4 x + e_5 \varepsilon^{F\prime} - e_6 \varepsilon^{x\prime} + e_7 \varepsilon^{D\prime}. \qquad \text{(A.15)}
$$

We can further simplify this expression by substituting out $\hat{x}$:

$$
\begin{aligned}
p_F F - p_x x &= p_F \hat{F} - p_x \hat{x} \\
\hat{x} &= \frac{p_F}{p_x} \hat{F} - \frac{p_F}{p_x} F + x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
D' + P' &= a + e_1 \hat{F} + e_2 \left(\frac{p_F}{p_x} \hat{F} - \frac{p_F}{p_x} F + x\right) + e_3 F - e_4 x + e_5 \varepsilon^{F\prime} - e_6 \varepsilon^{x\prime} \\
&\quad + e_7 \varepsilon^{D\prime} + e_8 \varepsilon^{S\prime} \\
&= a + \left(e_1 + e_2 \frac{p_F}{p_x}\right) \hat{F} + \left(e_3 - e_2 \frac{p_F}{p_x}\right) F + (-e_4 + e_2) x \\
&\quad + e_5 \varepsilon^{F\prime} - e_6 \varepsilon^{x\prime} + e_7 \varepsilon^{D\prime} + e_8 \varepsilon^{S\prime}.
\end{aligned}
$$

Given this expression, the conditional expectation for the informed is given by

$$
\begin{aligned}
E\left(D' + P' | \Omega^I\right) &= a + \left(e_1 + e_2 \frac{p_F}{p_x}\right) \hat{F} + \left(e_3 - e_2 \frac{p_F}{p_x}\right) F + (-e_4 + e_2) \hat{x} && \text{(A.16)} \\
Var\left(D' + P' | \Omega^I\right) &= e_5^2 \sigma_F^2 + e_6^2 \sigma_x^2 + e_7^2 \sigma_D^2 + e_8^2 \sigma_S^2 && \text{(A.17)}
\end{aligned}
$$

The conditional expectation for the uninformed is given by

$$E\left(D'+P'|\Omega^U\right) = a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)\hat{F}+\left(-e_4+e_2\right)\hat{x} \quad \text{(A.18)}$$

$$Var\left(D'+P'|\Omega^U\right) = H\Sigma H' + e_5^2\sigma_F^2+e_6^2\sigma_x^2+e_7^2\sigma_D^2+e_8^2\sigma_S^2, \quad \text{(A.19)}$$

where $H=\begin{bmatrix} e_3 & -e_4 \end{bmatrix} = \begin{bmatrix} \rho^F\left(1+p_F+p_{\hat{F}}\left(Y_1p_F+Y_2+Y_3\right)\right) & -\rho^x\left(p_{\hat{F}}Y_1p_x+p_x\right) \end{bmatrix}$; Σ solves equation A.2.

Now the market clearning is given by

$$\lambda D^I+\left(1-\lambda\right)D^U=x.$$

Plug in the demand function (A.3):

$$\lambda\frac{E\left[D'+P'|\Omega^I\right]-RP}{\alpha Var\left[D'+P'|\Omega^I\right]}+\left(1-\lambda\right)\frac{E\left[D'+P'|\Omega^I\right]-RP}{\alpha Var\left[D'+P'|\Omega^I\right]}=x.$$

Plug in the conditional expectation and variance from equations A.16 and A.18:

$$\lambda\frac{a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)F+\left(-e_4+e_2\right)x-RP}{\alpha V^I}$$

$$+\left(1-\lambda\right)\frac{a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)\hat{F}+\left(-e_4+e_2\right)x-RP}{\alpha V^U}=x$$

where $V^I=Var\left(D'+P'|\Omega^I\right);V^U=Var\left(D'+P'|\Omega^U\right).$

Rearranging and matching coefficients, we have three equations determining $p_{\hat{F}},p_F,p_x$:

$$\lambda\frac{e_1+e_2\frac{p_F}{p_x}}{\alpha V^I}+\left(1-\lambda\right)\frac{e_1+e_3}{\alpha V^U}=\left(\lambda\frac{R}{\alpha V^I}+\left(1-\lambda\right)\frac{R}{\alpha V^U}\right)p_{\hat{F}} \quad \text{(A.20)}$$

$$\lambda\frac{e_3-e_2\frac{p_F}{p_x}}{\alpha V^I}=\left(\lambda\frac{R}{\alpha V^I}+\left(1-\lambda\right)\frac{R}{\alpha V^U}\right)p_F \quad \text{(A.21)}$$

$$\lambda\frac{-e_4+e_2}{\alpha V^I}+\left(1-\lambda\right)\frac{-e_4+e_2}{\alpha V^U}-1=-\left(\lambda\frac{R}{\alpha V^I}+\left(1-\lambda\right)\frac{R}{\alpha V^U}\right)p_x \quad \text{(A.22)}$$

Note that we only need to keep track of one of $p_{\hat{F}}$ and p_F. To see this, sum equations A.20 and A.21:

$$\begin{aligned} e_1+e_3 &= \left(p_{\hat{F}}+p_F\right)R \\ p_{\hat{F}}\rho^F\left(1-Y_1p_F-Y_2\right)+\rho^F\left(1+p_F+p_{\hat{F}}\left(Y_1p_F+Y_2\right)\right) &= \left(p_{\hat{F}}+p_F\right)R \\ p_{\hat{F}}\rho^F+\rho^F+\rho^Fp_F &= \left(p_{\hat{F}}+p_F\right)R \end{aligned}$$

$$p_{\hat{F}} + p_F = \frac{\rho^F}{R - \rho^F}. \tag{A.23}$$

Thus, we only need to know, say, p_F, and we can deduce $p_{\hat{F}} = \frac{\rho^F}{R-\rho^F} - p_F$. Hence, for any λ, the exogenous information steady state $\Phi(\lambda)$ is (Σ, p_F, p_x) characterized by equations A.2, A.21, and A.22.

Next, I show that when $\lambda = 0$, the system of equations A.2, A.21, and A.22 can be explicitly solved. First, note that by equation A.21, $p_F = 0$. Second, by A.23, $p_{\hat{F}} = \frac{\rho^F}{R-\rho^F}$. To find p_x, rearrange A.22:

$$\begin{aligned}
\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda)\frac{1}{\alpha V^U}\right](-e_4 + e_2 + Rp_x) &= 1 \\
\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda)\frac{1}{\alpha V^U}\right]\left(-\rho^x\left(p_{\hat{F}} Y_1 p_x + p_x\right) + p_{\hat{F}} Y_1 p_x \rho^x + Rp_x\right) &= 1 \\
\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda)\frac{1}{\alpha V^U}\right](-\rho^x p_x + Rp_x) &= 1
\end{aligned}$$

$$(R - \rho^x)\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda)\frac{1}{\alpha V^U}\right] p_x = 1. \tag{A.24}$$

Next, we need to look for expressions of V^I, V^U. To do so, we need to first find the matrix

$$\Sigma = \begin{bmatrix} Var\left(F|\Omega^U\right) & Cov\left(F, x|\Omega^U\right) \\ Cov\left(F, x|\Omega^U\right) & Var\left(x|\Omega^U\right) \end{bmatrix}.$$

Note that everyone, including the uninformed, observes the price signal $S^P = p_F F - p_x x$. Thus

$$Cov\left(F, x|\Omega^U\right) = Cov\left(F, \frac{p_F F - S^P}{p_x}|\Omega^U\right) = \frac{p_F}{p_x} Var\left(F|\Omega^U\right)$$

$$Var\left(x|\Omega^U\right) = Var\left(\frac{p_F F - S^P}{p_x}|\Omega^U\right) = \left(\frac{p_F}{p_x}\right)^2 Var\left(F|\Omega^U\right).$$

Thus, we only need to find $Var\left(F|\Omega^U\right)$ to determine Σ. Denote $Var\left(F|\Omega^U\right) = \Sigma_F$.

Thus,

$$\Sigma = \begin{bmatrix} 1 & \frac{p_F}{p_x} \\ \frac{p_F}{p_x} & \left(\frac{p_F}{p_x}\right)^2 \end{bmatrix} \Sigma_F$$

Then,

$$A\Sigma A' = \begin{bmatrix} \left(\rho^F\right)^2 & \rho^F \rho^x \frac{p_F}{p_x} \\ \rho^F \rho^x \frac{p_F}{p_x} & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \end{bmatrix} \Sigma_F$$

$$\begin{aligned}
A\Sigma A' + Q &= \begin{bmatrix} \left(\rho^F\right)^2 & \rho^F \rho^x \frac{p_F}{p_x} \\ \rho^F \rho^x \frac{p_F}{p_x} & \left(\rho^x\right)^2 \left(\frac{p_F}{p_x}\right)^2 \end{bmatrix} \Sigma_F + \begin{bmatrix} \sigma_F^2 & \\ & \sigma_x^2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \\ \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & \left(\rho^x\right)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2 \end{bmatrix}
\end{aligned}$$

Note that

$$GA\Sigma A'G' + GQG' + R = \begin{bmatrix} \left(p_x\right)^2 \sigma_x^2 & 0 & 0 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix}$$

when $p_F = 0$. Now we can write out equation A.2 explicitly, which, when $\lambda = 0$ and thus $p_F = 0$, is reduced to

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Sigma_F &= \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \\ \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & \left(\rho^x\right)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2 \end{bmatrix} - \\
& \begin{bmatrix} 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ -p_x \sigma_x^2 & 0 & 0 \end{bmatrix} \\
& \begin{bmatrix} \left(p_x\right)^2 \sigma_x^2 & 0 & 0 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix}^{-1} \\
& \begin{bmatrix} 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & 0 & 0 \end{bmatrix}'
\end{aligned}$$

The determinant when $\lambda = 0$ is

$$\Theta = \left|\left[GA\Sigma A'G' + GQG' + R\right]\right| = \left(p_x\right)^2 \sigma_x^2 \left[\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)^2\right] \tag{A.25}$$

Thus,

$$\begin{aligned}
\left[GA\Sigma A'G' + GQG' + R\right]^{-1} &= \begin{bmatrix} \left(p_x\right)^2 \sigma_x^2 & 0 & 0 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix}^{-1} \\
&= \frac{1}{\Theta} \begin{bmatrix} \frac{\Theta}{\left(p_x\right)^2 \sigma_x^2} & 0 & 0 \\ 0 & \left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) & \left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right) \\ 0 & \left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right) & \left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\left(p_x\right)^2 \sigma_x^2} & 0 & 0 \\ 0 & \frac{\left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right)}{\Theta} & -\frac{\left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)}{\Theta} \\ 0 & -\frac{\left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)}{\Theta} & \frac{\left(p_x\right)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right)}{\Theta} \end{bmatrix} \text{ when } \lambda = 0
\end{aligned}$$

Plug in these terms and pick the first entry of the matrices. One obtains that Σ_F must solve the following equation when $\lambda = 0$:

$$\Sigma_F = \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \frac{\left(\sigma_S^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)^2}{\left[\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)^2\right]} \tag{A.26}$$

Denote the solution to this equation Σ_0 (we rule out negative roots). Intuitively, Σ_0 is the conditional mean of F for the uninformed when there are no informed investors.

We still need to obtain some expression for Y_1,Y_2, and Y_3 when $\lambda = 0$. Remember that they are entries of the first column of matrix $[A\Sigma A'G' + QG'][GA\Sigma A'G' + GQG' + R]^{-1}$.

We can take out each element of the first row of the matrix:

$$\theta_1 = Y_{1|\lambda=0} = 0 \tag{A.27}$$

$$\theta_2 = Y_{2|\lambda=0} = \frac{\sigma_S^2\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right]}{\left[\left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right)^2\right]} \tag{A.28}$$

$$\theta_3 = Y_{3|\lambda=0} = \frac{\sigma_D^2\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right]}{\left[\left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right)^2\right]} \tag{A.29}$$

Denote

$$\theta_0 = \theta_2 + \theta_3 \tag{A.30}$$

.

Crucially, note that

$$\begin{aligned}
\theta_0 &= \theta_2 + \theta_3 \\
&= \frac{\left(\sigma_D^2 + \sigma_S^2\right)\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right]}{\left[\left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right)^2\right]} \\
&= \frac{\left(\sigma_D^2 + \sigma_S^2\right)\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right]}{\left[\left(\sigma_D^2 + \sigma_S^2\right)\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right] + \sigma_D^2\sigma_S^2\right]} \in [0,1]
\end{aligned}$$

Now we are ready to derive expressions for V^I and V^U :

$$\begin{aligned}
V^I &= e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2 + e_8^2\sigma_S^2 \\
&= \left(1 + p_F + p_{\hat{F}}\left(Y_1 p_F + Y_2 + Y_3\right)\right)^2 \sigma_F^2 + \left(-\left(p_{\hat{F}} Y_1 p_x + p_x\right)\right)^2 \sigma_x^2 + \left(p_{\hat{F}} Y_2 + 1\right)^2 \sigma_D^2 + \left(p_{\hat{F}} Y_3\right)^2 \sigma_S^2 \\
&\to \left(1 + p_{\hat{F}}\left(Y_2 + Y_3\right)\right)^2 \sigma_F^2 + p_x^2\sigma_x^2 + \left(p_{\hat{F}} Y_2 + 1\right)^2 \sigma_D^2 + \left(p_{\hat{F}} Y_3\right)^2 \sigma_S^2 \\
&= \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \sigma_F^2 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2 + p_x^2\sigma_x^2
\end{aligned}$$

$$\begin{aligned} V^U &= H\Sigma H' + e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2 + e_8^2\sigma_S^2 \\ &\to \left[\begin{array}{cc} \rho^F\left(1+p_{\hat{F}}\left(Y_2+Y_3\right)\right) & -\rho^x p_x \end{array}\right]\left[\begin{array}{cc} \Sigma_0 & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} \rho^F\left(1+p_{\hat{F}}\left(Y_2+Y_3\right)\right) & -\rho^x p_x \end{array}\right]' \\ &\quad + \left(1+\frac{\rho^F}{R-\rho^F}\frac{\left(\rho^F\right)^2\Sigma_0+\sigma_F^2}{\sigma_D^2+\left(\rho^F\right)^2\Sigma_0+\sigma_F^2}\right)^2\left[\sigma_F^2+\sigma_D^2\right] + p_x^2\sigma_x^2 + \left(p_{\hat{F}}Y_3\right)^2\sigma_S^2 \\ &= \left(\rho^F\right)^2\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\Sigma_0 + \left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\sigma_F^2 + \left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2 + \left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2 + p_x^2\sigma_x^2 \end{aligned}$$

Plug expression for V^I and V^U back into equation A.24 and take $\lambda \to 0$, rearranging,

$$\alpha\sigma_x^2 p_x^2 - \left(R-\rho^x\right)p_x + \alpha\left(\left(\rho^F\right)^2\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\Sigma_0 + \left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\sigma_F^2 + \left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2 + \left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2\right)$$

Thus, p_x is well defined if and only if

$$\left(R-\rho^x\right)^2 - 4\alpha^2\sigma_x^2\left(\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\left[\left(\rho^F\right)^2\Sigma_0+\sigma_F^2\right] + \left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2 + \left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2\right) > 0$$

Denote $\Delta = \left(R-\rho^x\right)^2 - 4\alpha^2\sigma_x^2\left(\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\left[\left(\rho^F\right)^2\Sigma_0+\sigma_F^2\right] + \left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2 + \left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2\right)$, then at high volatility equilibria:

$$p_x = \frac{R-\rho^x+\sqrt{\Delta}}{2\alpha\sigma_x^2} \tag{A.31}$$

By continuity, when λ is very small, real roots to p_x still exists.

Auxiliary Results

Derivative of an inverse matrix

Proposition A.0.1 *Let A be a nonsingular, $m \times m$ matrix whose elements are functions of the scalar parameter α. Then:*

$$\frac{\partial A^{-1}}{\partial\alpha} = -A^{-1}\frac{\partial A}{\partial\alpha}A^{-1}$$

Proof. The definition of inverse is

$$A^{-1}A = I$$

Differentiating this expression with respect to α and rearranging, one obtains the result. ■

A.0.1 Proof of lemma 1.3.3

The proof is organized into two parts. The first part proves that the value of information is the ratio of uncertainty faced by the uninformed and informed and derives an exact formula for $\pi\left(\lambda\right)$. The second part derives the derivative $\pi'\left(\lambda\right)$ when $\lambda \to 0$.

Step 1: Deriving formula of $\pi(\lambda)$ We will show that, at the exogenous-information steady state $\Phi(\lambda)$, the value of information

$$\pi(\lambda) = \frac{W^U}{W^I} = \sqrt{\frac{Var(P' + D'|\Omega^U)}{Var(P' + D'|\Omega^I)}}$$

This is an extension of Theorem 2 in [1]. Note that P' denotes the next period equilibrium price whereas P denotes current price. Simplify agents' budget constraint: $c(s') = (D' + P' - RP)e$. Plugging into the utility function, we obtain the expected utility of each type of agent after the market opens:

$$W^i(P) = \max_e \int_{s'} U((D' + P' - RP)e)dH(s'|\Omega^i)$$

Given CARA utility:

$$\begin{aligned}
W^i(P) &= \max_e \int_{s'} U((D' + P' - RP)e)dH(s'|\Omega^i) \\
&= \max_e \int_{s'} -e^{(-\alpha((D'+P'-RP)e))}dH(s'|\Omega^i) \\
&= \max_e -\exp[E[-\alpha((D' + P' - RP)e)|I^i] + \frac{1}{2}Var(-\alpha((D' + P' - RP)e)|\Omega^i)] \\
&= \max_e -\exp[-\alpha(E[D' + P' - RP|\Omega^i]e - \frac{1}{2}\alpha e^2 Var(D' + P' - RP|\Omega^i))] \qquad \text{(A.32)}
\end{aligned}$$

Hence, maximizing over the objective function is equivalent to maximizing

$$\max_e E[D' + P' - RP|\Omega^i]e - \frac{1}{2}\alpha e^2 Var(D' + P' - RP|\Omega^i)$$

Solve for optimal s^*:

$$e^* = \frac{E[D' + P' - RP|\Omega^i]}{\alpha Var(D' + P' - RP|\Omega^i)}$$

Plug back into the original objective function:

$$\begin{aligned}
W^i(P) &= -\exp[-\frac{1}{2}\alpha\frac{(E[D' + P' - RP|\Omega^i])^2}{\alpha Var(D' + P' - RP|\Omega^i)}] \\
&= -\exp[-\frac{1}{2}\frac{(E[D' + P'|\Omega^i] - RP)^2}{Var(D' + P'|\Omega^i)}] \qquad \text{(A.33)}
\end{aligned}$$

where the second equation follows because P is realized at this stage. Let

$$h = Var(D' + P'|\Omega^U) - Var(D' + P'|\Omega^I) > 0$$

The reason why it is greater than 0 is that the uninformed have residual uncertainty over the current F whereas the informed are perfectly informed about F. Taking the conditional expectation of the informed $W_I(P)$ of the uninformed agents' information set:

$$\begin{aligned} E[W^i(P)|\Omega^U] &= E[-e^{-\frac{1}{2}\frac{(E[D'+P'|\Omega^I]-RP)^2}{Var(D'+P'|\Omega^I)}}|\Omega^U] \\ &= E[-e^{-\frac{1}{2}\frac{(E[D'+P'|\Omega^I]-RP)^2}{h}\frac{h}{Var(D'+P'|\Omega^I)}}|\Omega^U] \\ &= E[-e^{-\frac{1}{2}\frac{h}{Var(D'+P'|\Omega^I)}z^2}|\Omega^U], \end{aligned} \tag{A.34}$$

where $z = \frac{(E[D'+P'|\Omega^I]-RP)}{\sqrt{h}}$.

Thus, by the moment-generating function of a noncentral chi-squared distribution (formula $A21$ of [1]):

$$\begin{aligned} E[W^i(P)|\Omega^U] &= \frac{1}{\sqrt{1+\frac{h}{Var(D'+P'|\Omega^I)}}}\exp\Big(\frac{-E[z|\Omega^U]^2\frac{1}{2}\frac{h}{Var(D'+P'|\Omega^I)}}{1+\frac{h}{Var(D'+P'|\Omega^I)}}\Big) \\ &= \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|I^U)}}\exp\Big(\frac{-E[z|\Omega^U]^2\frac{1}{2}\frac{h}{Var(D'+P'|\Omega^i)}}{1+\frac{h}{Var(D'+P'|\Omega^i)}}\Big) \\ &= \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^U)}}\exp\Big(\frac{-\frac{1}{2}(E[D'+P'|\Omega^U]-RP)^2}{Var(D'+P'|\Omega^U)}\Big) \\ &= \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^U)}}W_U(P) \end{aligned}$$

Integrating on both sides with respect to the current state s, one gets:

$$W_I = \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^U)}}W_U \tag{A.35}$$

Thus,

$$\pi(\lambda) = \frac{W_U}{W_I} = \sqrt{\frac{Var(D'+P'|\Omega^U)}{Var(D'+P'|\Omega^I)}}$$

Now we know that

$$D'+P' = a + e_1\hat{F} + e_2\hat{x} + e_3F - e_4x + e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime}, \tag{A.36}$$

where expressions of e_i are given by A.6 through A.13.

Thus, by equation A.17 and A.19:

$$\begin{aligned} \pi(\lambda) = \frac{W_U}{W_I} &= \sqrt{\frac{Var(D'+P'|\Omega^U)}{Var(D'+P'|\Omega^I)}} \\ &= \sqrt{\frac{H\Sigma H' + e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2}{e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2}} \end{aligned} \tag{A.37}$$

Note that all the objects in A.37: H, Σ, e_5, e_6, e_7, are functions of Σ_F, p_F, p_x, which are ultimately implicit functions of λ, determined by equations A.2, A.21, and A.22. I omit the dependence here just to ease notation.

Step 2: Evaluate $\pi'(\lambda)$ when $\lambda \to 0$ First we represent equations A.2, A.21, and A.22 as

$$\begin{aligned} G_1(\Sigma_F, p_F, p_x) &= 0 \\ G_2(\Sigma_F, p_F, p_x, \lambda) &= 0 \\ G_3(\Sigma_F, p_F, p_x, \lambda) &= 0 \end{aligned}$$

from which we can derive the derivates $\frac{\partial \Sigma_F}{\partial \lambda}$, $\frac{\partial p_F}{\partial \lambda}$ and $\frac{\partial p_x}{\partial \lambda}$ by implicit differentiation. To begin, note that function G_1 does not contain λ. Therefore, we can think of Σ_F as an implicit function of p_F and p_x and evaluate the implicit derivative $\frac{\partial \Sigma_F}{\partial p_F}$ and $\frac{\partial \Sigma_F}{\partial p_x}$.We can show the following important result: $\frac{\partial \Sigma_F}{\partial p_F} = \frac{\partial \Sigma_F}{\partial p_x} = 0$:

$\mathrm{G}_1(\Sigma_F, p_F, p_x)$

$$= \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 p_F^2 \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[p_F^2 \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - p_x^2 \sigma_x^2\right]}{\sigma_D^2 \left[p_F^2 (\rho^F - \rho^x)^2 \Sigma_F + p_F^2 \sigma_F^2 + p_x^2 \sigma_x^2\right] + \sigma_F^2 (\rho^F)^2 p_F^2 \Sigma_F + (\rho^F)^2 p_x^2 \sigma_x^2 \Sigma_F + p_x^2 \sigma_x^2 \sigma_F^2} - \Sigma_F$$

$$= \left[\left(\rho^F\right)^2 - 1\right] \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 \frac{p_F^2}{p_x^2} \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[\frac{p_F^2}{p_x^2} \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - \sigma_x^2\right]}{\sigma_D^2 \left[\frac{p_F^2}{p_x^2} (\rho^F - \rho^x)^2 \Sigma_F + \frac{p_F^2}{p_x^2} \sigma_F^2 + \sigma_x^2\right] + \sigma_F^2 (\rho^F)^2 \frac{p_F^2}{p_x^2} \Sigma_F + (\rho^F)^2 \sigma_x^2 \Sigma_F + \sigma_x^2 \sigma_F^2}$$

$$= \left[\left(\rho^F\right)^2 - 1\right] \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 u \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[u \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - \sigma_x^2\right]}{\sigma_D^2 \left[u (\rho^F - \rho^x)^2 \Sigma_F + u \sigma_F^2 + \sigma_x^2\right] + \sigma_F^2 (\rho^F)^2 \Sigma_F u + (\rho^F)^2 \sigma_x^2 \Sigma_F + \sigma_x^2 \sigma_F^2}$$

where $\mathrm{u} = \left[\frac{p_F^2}{p_x^2}\right]^2$

Define a new function:

$$N_1(\Sigma_F, u) = \left[\left(\rho^F\right)^2 - 1\right] \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 u \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[u \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - \sigma_x^2\right]}{\sigma_D^2 \left[u \left(\rho^F - \rho^x\right)^2 \Sigma_F + u \sigma_F^2 + \sigma_x^2\right] + \sigma_F^2 \left(\rho^F\right)^2 \Sigma_F u + \left(\rho^F\right)^2 \sigma_x^2 \Sigma_F + \sigma_x^2 \sigma_F^2}$$

Thus,

$$G_1(\Sigma_F, p_F, p_x) = N_1\left(\Sigma_F, \frac{p_F^2}{p_x^2}\right)$$

By implicit differentiation:

$$\frac{\partial \Sigma_F}{\partial p_F} = -\frac{\frac{\partial G_1}{\partial p_F}}{\frac{\partial G_1}{\partial \Sigma_F}} - \frac{\frac{\partial N_1}{\partial u} \frac{\partial u}{\partial p_F}}{\frac{\partial N_1}{\partial \Sigma_F}} = -\frac{\frac{\partial N_1}{\partial u}}{\frac{\partial N_1}{\partial \Sigma_F}} 2 \frac{p_F}{p_x^2} \to 0$$

Likewise,

$$\frac{\partial \Sigma_F}{\partial p_x} \to 0$$

Now, G_2 and G_3 are given by

$$G_2 = \lambda \frac{e_3 - e_2 \frac{p_F}{p_x}}{\alpha V^I} - \left(\lambda \frac{R}{\alpha V^I} + (1-\lambda)\frac{R}{\alpha V^U}\right) p_F = 0$$

$$G_3 = \left[\lambda \frac{1}{\alpha V^I} + (1-\lambda)\frac{1}{\alpha V^U}\right]\left[-e_4 + e_2 + R p_x\right] - 1 = 0$$

To derive $\frac{\partial p_F}{\partial \lambda}$ and $\frac{\partial p_x}{\partial \lambda}$, define $\pi = \frac{V^U}{V^I}$. Substitute in π whenever possible. Then,

$$\begin{aligned}
G_2(\lambda, p_F, p_x, \pi) &= \lambda \frac{e_3 - e_2 \frac{p_F}{p_x}}{\alpha}\pi - \left(\lambda \frac{1}{\alpha}\pi + (1-\lambda)\frac{1}{\alpha}\right) R p_F = 0 \\
G_3(\lambda, p_F, p_x, \pi) &= \left[\frac{\lambda}{\alpha}\pi + (1-\lambda)\frac{1}{\alpha}\right]\left[-e_4 + e_2 + R p_x\right] - V^U = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial G_2}{\partial \lambda} &= \frac{e_3 - e_2 \frac{p_F}{p_x}}{\alpha}\pi \to \frac{\rho^F\left(1 + \frac{\rho^F}{R-\rho^F}\theta_0\right)}{\alpha}\pi \\
\frac{\partial G_2}{\partial p_F} &= \lambda \frac{\frac{\partial e_3 - e_2 \frac{p_F}{p_x}}{\partial p_F}}{\alpha}\pi - \left(\lambda \frac{R}{\alpha}\pi + (1-\lambda)\frac{R}{\alpha}\right) = -\frac{R}{\alpha} \\
\frac{\partial G_2}{\partial p_x} &= 0 \\
\frac{\partial G_2}{\partial \pi} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial G_3}{\partial \lambda} &= \left[\frac{1}{\alpha}\pi - \frac{1}{\alpha}\right]\left[-e_4 + e_2 + R p_x\right] \to \frac{1}{\alpha}\left[\pi - 1\right]\left[R - \rho^x\right] p_x \\
\frac{\partial G_3}{\partial p_F} &= \left[\frac{\lambda}{\alpha}\pi + (1-\lambda)\frac{1}{\alpha}\right]\left[\frac{\partial\left(-e_4 + e_2\right)}{\partial p_F}\right] - \frac{\partial V^U}{\partial p_F} = -\frac{1}{\alpha}\rho_x p_x \frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F} - \frac{\partial V^U}{\partial p_F} \\
\frac{\partial G_3}{\partial p_x} &= \left[\frac{\lambda}{\alpha}\pi + (1-\lambda)\frac{1}{\alpha}\right]\left[\frac{\partial\left(-e_4 + e_2\right)}{\partial p_x} + R\right] - \frac{\partial V^U}{\partial p_x} = \frac{1}{\alpha}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x} \\
\frac{\partial G_3}{\partial \pi} &= 0
\end{aligned}$$

Total differentiation of G_2 and G_3 gives

$$\begin{bmatrix} \frac{\partial G_2}{\partial p_F} + \frac{\partial G_2}{\partial \pi}\frac{\partial \pi}{\partial p_F} & \frac{\partial G_2}{\partial p_x} + \frac{\partial G_2}{\partial \pi}\frac{\partial \pi}{\partial p_x} \\ \frac{\partial G_3}{\partial p_F} + \frac{\partial G_3}{\partial \pi}\frac{\partial \pi}{\partial p_F} & \frac{\partial G_3}{\partial p_x} + \frac{\partial G_3}{\partial \pi}\frac{\partial \pi}{\partial p_x} \end{bmatrix}\begin{bmatrix} \frac{\partial p_F}{\partial \lambda} \\ \frac{\partial p_x}{\partial \lambda} \end{bmatrix} = -\begin{bmatrix} \frac{\partial G_2}{\partial \lambda} \\ \frac{\partial G_3}{\partial \lambda} \end{bmatrix}$$

Substituting in each term gives

$$\begin{bmatrix} -\frac{R}{\alpha} & 0 \\ -\frac{1}{\alpha}\rho_x p_x \frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F} - \frac{\partial V^U}{\partial p_F} & \frac{1}{\alpha}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x} \end{bmatrix}\begin{bmatrix} \frac{\partial p_F}{\partial \lambda} \\ \frac{\partial p_x}{\partial \lambda} \end{bmatrix} = -\begin{bmatrix} \frac{\rho^F\left(1 + \frac{\rho^F}{R-\rho^F}\theta_0\right)}{\alpha}\pi \\ \frac{1}{\alpha}\left[\pi - 1\right]\left[R - \rho^x\right] p_x \end{bmatrix}$$

$$\frac{\partial p_F}{\partial \lambda} = \frac{\rho^F\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)}{R}\pi \tag{A.38}$$

$$\frac{\partial p_x}{\partial \lambda} = \frac{-\frac{1}{\alpha}\left[\pi-1\right]\left[R-\rho^x\right]p_x+\left(\frac{1}{\alpha}\rho_x p_x\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F}+\frac{\partial V^U}{\partial p_F}\right)\frac{\partial p_F}{\partial \lambda}}{\frac{1}{\alpha}\left[-\rho^x+R\right]-\frac{\partial V^U}{\partial p_x}}$$

We can also evaluate

$$\begin{aligned}\frac{\partial \Sigma_F}{\partial \lambda} &= \frac{\partial \Sigma_F}{\partial p_F}\frac{\partial p_F}{\partial \lambda}+\frac{\partial \Sigma_F}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\\ &= 0\frac{\partial p_F}{\partial \lambda}+0\frac{\partial p_x}{\partial \lambda}=0\end{aligned}$$

We are ready to evaluate $\pi'(\lambda)$ when $\lambda \to 0$.

Define

$$\Pi\left(\Sigma_F, p_F, p_x\right)=\frac{H\Sigma H+e_5^2\sigma_F^2+e_6^2\sigma_x^2+e_7^2\sigma_D^2}{e_5^2\sigma_F^2+e_6^2\sigma_x^2+e_7^2\sigma_D^2}$$

Then,

$$\pi(\lambda)=\sqrt{\Pi\left(\Sigma_F(\lambda), p_F(\lambda), p_x(\lambda)\right)}$$

$$\begin{aligned}\pi'(\lambda) &= \frac{1}{2}\Pi^{-\frac{1}{2}}\left[\frac{\partial \Pi}{\partial \Sigma_F}\frac{\partial \Sigma_F}{\partial \lambda}+\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda}+\frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\right]\\ &= \frac{1}{2}\Pi^{-\frac{1}{2}}\left[\frac{\partial \Pi}{\partial \Sigma_F}0+\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda}+\frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\right]\\ &= \frac{1}{2}\Pi^{-\frac{1}{2}}\left[\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda}+\frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\right]\end{aligned}$$

Thus, the sign of $\pi'(\lambda)$ only depends on the sign of $\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda}+\frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}$.

Thus, $\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda}+\frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}>0$ if and only if

$$\left(1-\theta_0\right)\rho^F>\rho^x+\phi(\rho^F,\rho^x,\sigma_x^2,\sigma_F^2,\sigma_D^2,\sigma_S^2,\alpha,R) \tag{A.39}$$

Appendix B

Appendix to Chapter 2

B.0.1 Proof of the main theorem

The proof consists of two parts. In the first part, I prove that there exists multiple steady states. In the second part, I proved that some (more than one) of these steady states are locally stable.

To prove the first part, we first need to solve for the unique unconstrained steady state dropping the borrowing constraint. The equations are given by:

$$\omega h_0^{-\sigma}\left(c-\chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma}+\beta p = p \tag{B.1}$$

$$\beta\left[A\alpha k^{\alpha-1}l^{1-\alpha}+(1-\delta)\right] = 1 \tag{B.2}$$

$$c+\delta k = Ak^{\alpha}l^{1-\alpha} \tag{B.3}$$

$$l^{\frac{1}{\nu}} = w \tag{B.4}$$

$$l = \left(\frac{\gamma A}{w}\right)^{\frac{1}{\alpha}}k \tag{B.5}$$

By the second equation we obtain the capital-labor ratio:

$$\frac{k}{l}=\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}} \tag{B.6}$$

Plug it into equation B.5, we obtain the steady state level of wage. call it w_{ss}:

$$\begin{aligned} w^{\frac{1}{\alpha}} &= (\gamma A)^{\frac{1}{\alpha}}\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}} \\ w_{ss} &= \gamma A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}} \end{aligned}$$

Plug w_{ss} back to equation B.4, we obtain the steady state level of labor, l_{ss}:

$$l_{ss}=w_{ss}^{\nu}=\left[(1-\alpha)A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right]^{\nu}$$

Plug this into equation B.6, we obtain the steady state level of capital

$$\begin{aligned} k_{ss} &= \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}} l \\ &= \left((1-\alpha)A\right)^{\nu}\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1+\alpha\nu}{\alpha-1}} \end{aligned}$$

And steady state level of consumption is given by:

$$\begin{aligned} c_{ss} &= Ak_{ss}^{\alpha}l_{ss}^{1-\alpha}-\delta k_{ss} \\ &= \left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}-\delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right] l_{ss} \\ &= \left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}-\delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right]\left[(1-\alpha)A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right]^{\nu} \end{aligned}$$

Lastly, the steady state land price p_{ss} is given by:

$$\begin{aligned} (1-\beta)p_{ss} &= \omega h_0^{-\sigma}\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma} \\ p_{ss} &= \frac{\omega h_0^{-\sigma}\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma}}{(1-\beta)} \end{aligned}$$

And as this is a closed economy, agents ' bond holding is equal to 0

$$b_{ss}=0$$

Given objects $(w_{ss}, l_{ss}, k_{ss}, c_{ss}, p_{ss})$ we define

$$\xi_{ss}=\frac{w_{ss}l_{ss}-\kappa k_{ss}}{p_{ss}h_0}>0$$

For κ sufficiently small it is feasible.

Next, we would like to show the following: for sufficiently small κ, given ξ_{ss}, there exists multiple steady states in benchmark model. To show this, note that the unconstrained steady states is automatically a steady state because by definition of ξ_{ss}, it satisfies the borrowing constraint. Thus we only need to show that there exists another steady state and we are done. The system characterizing the steady state is given, as in the main text, by

$$\begin{aligned} \omega h_0^{-\frac{1}{\eta}}\left(c-\chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\frac{1}{\eta}}+\beta p+\left(\frac{(1-\alpha)Ak^{\alpha}(l)^{\gamma-1}}{w}-1\right)\xi p &= p \\ \beta\left[A\alpha k^{\alpha-1}(l)^{\gamma}+(1-\delta)+\left(\frac{(1-\alpha)Ak^{\alpha}(l)^{\gamma-1}}{w}-1\right)\kappa\right] &= 1 \\ c+\delta k &= Ak^{\alpha}(l)^{1-\alpha} \\ l^{\frac{1}{\nu}} &= w \\ \min\left(\frac{\xi_{ss}ph_0+\kappa k}{w}, \left(\frac{(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}}k\right) &= l \end{aligned}$$

The only difference is that we focus on the case where $\xi = \xi_{ss}$.

We wish to show that apart from $(w_{ss}, l_{ss}, k_{ss}, c_{ss}, p_{ss})$, there exists another set of variables that solve these equations. First of all, we conjecture, and later verify that at the other steady state

$$\frac{\xi_{ss} p h_0}{w} \leq \left(\frac{(1-\alpha) A}{w}\right)^{\frac{1}{\alpha}} k$$

Thus we end up with a differentiable system of equations

$$\begin{aligned}
\omega h_0^{-\sigma}\left(c-\chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma}+\beta p+\left(\frac{(1-\alpha) A k^{\alpha} l^{-\alpha}}{w}-1\right)\xi p &= p \\
\beta\left[A\alpha k^{\alpha-1} l^{1-\alpha}+(1-\delta)+\left(\frac{(1-\alpha) A k^{\alpha}(l)^{\gamma-1}}{w}-1\right)\kappa\right] &= 1 \\
c+\delta k &= A k^{\alpha}(l)^{1-\alpha} \\
l^{\frac{1}{\nu}} &= w \\
\frac{\xi_{ss} p h_0+\kappa k}{w} &= l
\end{aligned}$$

Also since we take κ to be sufficiently small, we can further simplify the system considerably by looking at the limiting case where $\kappa = 0$ (Rigorously speaking, we are interchanging limit and differentiation. To do so, we need to show that for a sequence of $\kappa_n \to 0$, the resulting sequence of functions $f(p; \kappa_n)$ converges uniformly to $f(p; 0)$. This is guaranteed by the fact that f is differntiable with respect to κ, and the derivative is bounded. See Walter Rudin's Principle of Mathematical Analysis, 3rd Edition, Theorem 7.17):

$$\omega h_0^{-\frac{1}{\eta}}\left(c-\chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma}+\beta p+\left(\frac{(1-\alpha) A k^{\alpha}(l)^{-\alpha}}{w}-1\right)\xi p = p \tag{B.7}$$

$$\beta\left[A\alpha k^{\alpha-1}(l)^{1-\alpha}+(1-\delta)\right] = 1 \tag{B.8}$$

$$c+\delta k = A k^{\alpha}(l)^{1-\alpha} \tag{B.9}$$

$$l^{\frac{1}{\nu}} = w \tag{B.10}$$

$$\frac{\xi_{ss} p h_0}{w} = l \tag{B.11}$$

From the second equation of the system, we have

$$l = \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{1-\alpha}} k$$

Thus by the fourth equation

$$\begin{aligned}
w &= l^{\frac{1}{\nu}} \\
&= \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{(1-\alpha)\nu}} k^{\frac{1}{\nu}}
\end{aligned}$$

We can substitute out w by plugging in the fifth equation

$$\frac{\xi_{ss} p h_0}{w} = l$$

We obtain

$$\begin{aligned}
\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{(1-\alpha)}} k &= (\xi_{ss} h_0 p)^{\frac{1}{1+\frac{1}{\nu}}} \\
k &= M p^{\frac{\nu}{1+\nu}}
\end{aligned}$$

Where M is a constant defined as

$$M = \frac{(\xi_{ss}h_0)^{\frac{1}{1+\frac{1}{\nu}}}}{\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{(1-\alpha)}}}$$

Also note that

$$\begin{aligned} l &= \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{1-\alpha}} k \\ &= \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{1-\alpha}} Mp^{\frac{\nu}{1+\nu}} \\ &= Np^{\frac{\nu}{1+\nu}} \end{aligned}$$

Where N is a constant defined as

$$N = M\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{1-\alpha}}$$

We also obtain wage:

$$\begin{aligned} w &= l^{\frac{1}{\nu}} \\ &= \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{(1-\alpha)\nu}} k^{\frac{1}{\nu}} \\ &= \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{(1-\alpha)\nu}} \left(Mp^{\frac{\nu}{1+\nu}}\right)^{\frac{1}{\nu}} \\ &= Qp^{\frac{1}{1+\nu}} \end{aligned}$$

Where Q is yet another constant defined as

$$Q = \left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{(1-\alpha)\nu}} M^{\frac{1}{\nu}}$$

Thus we can also express consumption as a function of land price p :

$$\begin{aligned} c &= Ak^{\alpha}l^{1-\alpha} - \delta k \\ &= A\left(Mp^{\frac{\nu}{1+\nu}}\right)^{\alpha}\left(Np^{\frac{\nu}{1+\nu}}\right)^{1-\alpha} - \delta Mp^{\frac{\nu}{1+\nu}} \\ &= \left[A(M)^{\alpha}(N)^{1-\alpha} - \delta M\right]p^{\frac{\nu}{1+\nu}} \end{aligned}$$

Plug everything into the equation B.4 and we have one equation with one unknown.

$$f(p) = \frac{(1-\omega)}{\omega}h_0^{-\sigma}\left(c - \chi\frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma} + \beta p + \left(\frac{(1-\alpha)Ak^{\alpha}(l)^{-\alpha}}{w} - 1\right)\xi p - p$$

Rearrange:

$$f(p) = \frac{(1-\omega)}{\omega}h_0^{-\sigma}\left(\left[A(M)^{\alpha}(N)^{1-\alpha} - \delta M\right]p^{\frac{\nu}{1+\nu}} - \chi\frac{N^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}p\right)^{\sigma} + \frac{(1-\alpha)A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{-\alpha}{1-\alpha}}}{Q}\xi p^{\frac{\nu}{1+\nu}} = 0 \tag{B.12}$$

We wish to show that equality B.12 has multiple roots. As argued previously, one root is p_{ss} as the unconstrained steady state is automatically a steady state by the choice of ξ_{ss}. Also note that $p = 0$ is a trivial steady state. Next we would like to show the following two statements

1. $f'(p_{ss}) > 0$
2. $f'(0) > 0$

If the above two statements are true, then by continuity of f and the intermediate value theorem, we obtain another nontrivial steady state $p \in (0, p_{ss})$

The derivative $f'(p)$ is given by:

$$\begin{aligned} f'(p) &= \sigma\frac{(1-\omega)}{\omega}h_0^{-\sigma}\left(\left[A(M)^{\alpha}(N)^{1-\alpha}-\delta M\right]p^{\frac{\nu}{1+\nu}}-\chi\frac{N^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}p\right)^{\sigma-1} \\ &\quad \left[\frac{\nu}{1+\nu}\left[A(M)^{\alpha}(N)^{1-\alpha}-\delta M\right]p^{\frac{\nu}{1+\nu}-1}-\chi\frac{N^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right] \\ &\quad +\frac{\nu}{1+\nu}\frac{(1-\alpha)A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{-\alpha}{1-\alpha}}}{Q}\xi p^{\frac{\nu}{1+\nu}-1}-(1-\beta+\xi) \end{aligned} \tag{B.13}$$

We want to show that the first statement holds: $f'(p_{ss}) > 0$. Plug $p = p_{ss}$ into equation B.14:

$$\begin{aligned} f'(p_{ss}) &= \sigma\frac{(1-\omega)}{\omega}h_0^{-\sigma}\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma-1}\left[\frac{\nu}{1+\nu}c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right]\frac{1}{p_{ss}} \\ &\quad +\frac{\nu}{1+\nu}\frac{(1-\alpha)A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{-\alpha}{1-\alpha}}}{Q}\xi p_{ss}^{\frac{-1}{1+\nu}}-(1-\beta+\xi) \\ &= \sigma\frac{(1-\omega)}{\omega}h_0^{-\sigma}\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma-1}\left[\frac{\nu}{1+\nu}c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right]\frac{1}{p_{ss}} \\ &\quad -\frac{1}{1+\nu}\xi-(1-\beta) \\ &= \sigma\frac{(1-\omega)}{\omega}h_0^{-\sigma}\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma-1}\left[\frac{\nu}{1+\nu}c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right]\frac{1-\beta}{\frac{(1-\omega)}{\omega}h_0^{-\frac{1}{\eta}}\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\frac{1}{\eta}}} \\ &\quad -\frac{1}{1+\nu}\xi-(1-\beta) \\ &= (1-\beta)\sigma\left(c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-1}\left[\frac{\nu}{1+\nu}c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right] \\ &\quad -\frac{1}{1+\nu}\xi-(1-\beta) \\ &= (1-\beta)\sigma\frac{\frac{\nu}{1+\nu}c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}}{c_{ss}-\chi\frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}}-\frac{1}{1+\nu}\xi-(1-\beta) \end{aligned}$$

Note that for $f'(p_{ss}) > 0$, the first term

$$(1-\beta)\,\sigma \frac{\frac{\nu}{1+\nu}c_{ss} - \chi \frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}}{c_{ss} - \chi \frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}}$$

must be positive and sufficiently big, as the next two terms are negative. Note that when $\nu \to \infty$,

$$f'(p_{ss}) = (1-\beta)(\sigma - 1)$$

Thus when $\nu \to \infty$, I only need $\sigma > 1$ to guarantee that $f'(p_{ss}) > 0$.

On the other hand, when σ is sufficiently big, I only need $v > \nu_0$ where ν_0 is given by:

$$\nu_0 = \chi \frac{1-\alpha}{1 - \frac{\alpha\delta}{\frac{1}{\beta} - 1 + \delta}}$$

When $\nu > \nu_0$,

$$\frac{\nu}{1+\nu}c_{ss} - \chi \frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} > 0$$

This guarantees that when $\sigma \to +\infty$

$$(1-\beta)\,\sigma \frac{\frac{\nu}{1+\nu}c_{ss} - \chi \frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}}{c_{ss} - \chi \frac{l_{ss}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}} \to +\infty$$

Thus,

$$f'(p_{ss}) > 0 \text{ as } \sigma \text{ sufficiently big}$$

In sum, for σ and ν sufficiently big, $f'(p_{ss}) > 0$.

Let's turn to the second statement $f'(0) > 0$

Take $p \to 0$ into equation B.14. We examine the behavior of each term separately. Note that the first term

$$\sigma \frac{(1-\omega)}{\omega} h_0^{-\sigma} \left(\left[A(M)^{\alpha}(N)^{1-\alpha} - \delta M \right] p^{\frac{\nu}{1+\nu}} - \chi \frac{N^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} p \right)^{\sigma-1}$$
$$\left[\frac{\nu}{1+\nu} \left[A(M)^{\alpha}(N)^{1-\alpha} - \delta M \right] p^{\frac{\nu}{1+\nu}-1} - \chi \frac{N^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right]$$

is of the same order of

$$p^{\frac{\nu}{1+\nu}\sigma - 1}$$

and thus $\to 0$ when σ is sufficiently big The second term

$$\frac{\nu}{1+\nu} \frac{(1-\alpha) A \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{-\alpha}{1-\alpha}}}{Q} \xi p^{\frac{\nu}{1+\nu} - 1} \to +\infty$$

as $p^{\frac{\nu}{1+\nu}-1} \to +\infty$

The third term is a constant.

$$-(1 - \beta + \xi)$$

Thus, for σ sufficiently big, the second term dominates and therefore

$$f'(0) > 0$$

Lastly, we need to verify that at the new steady state:

$$\frac{\xi_{ss} p h_0}{w} < \left(\frac{(1-\alpha) A}{w}\right)^{\frac{1}{\alpha}} k$$

Plug in k, w as a function of p derived from other equilibrium conditions:

$$\begin{aligned} k &= Mp^{\frac{\nu}{1+\nu}} \\ w &= Qp^{\frac{1}{1+\nu}} \end{aligned}$$

we obtain:

$$\frac{\xi_{ss} p h_0}{Qp^{\frac{1}{1+\nu}}} < \left(\frac{(1-\alpha) A}{Qp^{\frac{1}{1+\nu}}}\right)^{\frac{1}{\alpha}} Mp^{\frac{\nu}{1+\nu}}$$

Rearrange:

$$\frac{\xi_{ss} h_0}{Q} p^{\frac{\nu}{1+\nu}} \left(\frac{(1-\alpha) A}{Q}\right)^{\frac{1}{\alpha}} Mp^{\frac{\nu}{1+\nu}-\frac{1}{\alpha}\frac{1}{1+\nu}}$$

To do this comparison, first note that at $p = p_{ss}$, the equation is equalized:

$$\frac{\xi_{ss} p_{ss} h_0}{Qp_{ss}^{\frac{1}{1+\nu}}} = \left(\frac{(1-\alpha) A}{Qp_{ss}^{\frac{1}{1+\nu}}}\right)^{\frac{1}{\alpha}} Mp_{ss}^{\frac{\nu}{1+\nu}}$$

But for

$$p < p_{ss}$$

We have

$$\frac{\xi_{ss} h_0}{Q} p^{\frac{\nu}{1+\nu}} < \left(\frac{(1-\alpha) A}{Q}\right)^{\frac{1}{\alpha}} Mp^{\frac{\nu}{1+\nu}-\frac{1}{\alpha}\frac{1}{1+\nu}}$$

Because the left hand side decreases faster with p than the right hand side. Therefore we have

$$\frac{\xi_{ss} p h_0}{w} < \left(\frac{(1-\alpha) A}{w}\right)^{\frac{1}{\alpha}} k$$

At any steady state where $p < p_{ss}$. Therefore we have established that for $\xi = \xi_{ss}$ there exists multiple nontrivial steady states.

Next, we estabilish that for $\xi > \xi_{ss}$, but sufficiently close to ξ_{ss}, there exists multiple nontrivial steady states as well. This is obvious as $f(p)$ is continuous with respect to ξ. Therefore, for sufficiently small changes in ξ, we still have multiple nontrivial steady states.

To see this more precisely, we know that given $\xi = \xi_{ss}$, there exists $p_1 < p_2$ such that $f(p_1; \xi = \xi_{ss}) > 0$ and $f(p_2; \xi = \xi_{ss}) < 0$. By continuity of f with respect to ξ, a small change in ξ still preverves that $f(p_1; \xi) > 0$ and $f(p_2) < 0$.

This implies that multiple nontrivial steady states exist. Thus we can pick an interval U^{ξ} such that for any $\xi \in U^{\xi}$, multiple steady states exist.

Next, we would like to show that more than 1 steady states are locally stable. Pick a ξ such that $\xi > \xi_{ss}$. We need to show the following three lemmas to prove locally stability.

Lemma B.0.1 *The unconstrained steady state is locally stable*

Proof. *As the credit constraint is slack, the dynamic system is identical to that of a standard neoclassical model. Thus standard argument applies and therefore the steady state is locally stable. See SLP.* ■

Lemma B.0.2 *There exists* $0 < p < p_{ss}$ *such that* $f(p) = 0$ *and* $f'(p) < 0$

Proof. *we know that there exists two points* $p_1 < p_2$ *such that* $f(p_1) > 0$ *and* $f(p_2) < 0$*. By continuity, this implies that there exists at least one* $p \in (p_1, p_2)$ *such that* $f'(p) < 0$*. To see this, suppose the contrary, that for any* p *such that* $f(p) = 0, f'(p) > 0$*. Denote the set of steady state* $p : P = \{p; f(p) = 0\}$*. Pick* $p_{\inf} = \inf P$*. By continuity* $f(p_{\inf}) = 0$*. But we know that every* $f(p) = 0$ *has the property that* $f'(p) > 0$*. Thus* $f'(p_{\inf}) > 0$*. This implies that there exists* $p < p_{\inf}$ *such that* $f(p) = 0$ *by the fact that* $f(p_1) > 0$*. A contradiction. Therefore there must exist at least one* $p \in (p_1, p_2)$ *such that* $f'(p) < 0$*.* ■

The second lemma will be useful later. Right now let's write down the dynamic system around the constrained steady state:

$$\begin{aligned}
\omega h_0^{-\sigma}\left(c_t - \chi\frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{\sigma} + \beta p_{t+1} + \left(\frac{(1-\alpha)\,Ak^{\alpha}l^{-\alpha}}{w} - 1\right)\xi p_t &= p_t \\
\beta\left(c_{t+1} - \chi\frac{l_{t+1}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-\sigma}\left[A\alpha k_{t+1}^{\alpha-1}l_{t+1}^{1-\alpha} + (1-\delta)\right] &= \left(c_t - \chi\frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{-\sigma} \\
c_t + \delta k_t &= Ak_t^{\alpha}(l_t)^{1-\alpha} \\
l_t^{\frac{1}{\nu}} &= w_t \\
\frac{\xi p_t h_0}{w_t} &= l_t
\end{aligned}$$

From the last two equations we can substitute out l_t :

$$(\xi p_t h_0)^{\frac{\nu}{\nu+1}} = l_t$$

And we are left with a three-equation dynamic system with three state variables (c_t, k_t, p_t) :

$$\begin{aligned}
\omega h_0^{-\sigma}\left(c_t - \chi\frac{\xi p_t h_0}{1+\frac{1}{\nu}}\right)^{\sigma} + \beta p_{t+1} + \left(\frac{(1-\alpha)\,Ak^{\alpha}l^{-\alpha}}{w} - 1\right)\xi p_t &= p_t \\
\beta\left(c_{t+1} - \chi\frac{\xi p_{t+1} h_0}{1+\frac{1}{\nu}}\right)^{-\sigma}\left[A\alpha k_{t+1}^{\alpha-1}l_{t+1}^{1-\alpha} + (1-\delta)\right] &= \left(c_t - \chi\frac{\xi p_t h_0}{1+\frac{1}{\nu}}\right)^{-\sigma} \\
c_t + \delta k_t &= Ak_t^{\alpha}(\xi p_t h_0)^{\frac{\nu}{\nu+1}(1-\alpha)}
\end{aligned} \tag{B.14}$$

We denote $S = (c, k)$. And we can write this equation compactly as

$$\begin{bmatrix} p_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} F(p_t, S_t) \\ G(p_t, S_t) \end{bmatrix}$$

Where F, G are the transition matrices implicitly defined by the system B.14.

Linearize it around some steady state:

$$\begin{bmatrix} p_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} F_p & F_S \\ G_p & G_S \end{bmatrix}\begin{bmatrix} p_t \\ S_t \end{bmatrix}$$

As there is one predetermined variable, k, We need to show that the eigenvalues associated with the linearized transition matrix $\begin{bmatrix} F_p & F_S \\ G_p & G_S \end{bmatrix}$ has one and only one roots within the unit circle. Write down the characteristic root equation:

$$\delta(\lambda) = \begin{bmatrix} F_p - \lambda & F_S \\ G_p & G_S - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \end{bmatrix}$$

This system has three roots $\lambda_1, \lambda_2, \lambda_3$. We need to show that there is a unique root, call it λ_1, such that $|\lambda_1| < 1$.To do that, we need to show that:

1. $\delta(0) > 0$
2. $\delta(1) < 0$
3. $trace\left(\begin{bmatrix} F_p & F_S \\ G_p & G_S \end{bmatrix}\right) > 3$

See the online appendix for a detailed derivation of the three statements. The first two requirement guarantees that there exists a root $|\lambda_1| < 1$. The last requirement guarantees that only one root is inside the unit circle as

$$\lambda_1 + \lambda_2 + \lambda_3 = trace\left(\begin{bmatrix} F_p & F_S \\ G_p & G_S \end{bmatrix}\right)$$

Requirement 1 and 3 are automatically satisfied when σ and ν are sufficiently big. Requirement 2 is satisfied whenever $f'(p) < 0$. To see this note that the steady state equation is given by

$$f(p) = p - F(k(p), p)$$

Thus

$$f'(p) = 1 - F_k k_p - F_p$$

Where k_p is implicitly given by:

$$k - G(k, p) = 0$$

$$\begin{aligned} k_p - G_k k_p - G_p &= 0 \\ k_p &= [I - G_k]^{-1}(G_p) \end{aligned}$$

Thus

$$f'(p) = 1 - F_k [I - G_k]^{-1}(G_p) - F_p$$

Linearize the system:

$$\begin{bmatrix} p' \\ k' \end{bmatrix} = \begin{bmatrix} F_p & F_k \\ G_p & G_k \end{bmatrix} \begin{bmatrix} p \\ k \end{bmatrix}$$

Thus the characteristic roots are given by

$$\begin{aligned} \delta(\lambda) &= \begin{vmatrix} F_p - \lambda & F_k \\ G_p & G_k - \lambda I \end{vmatrix} \\ &= \det\left((F_p - \lambda) - F_k (G_k - \lambda I)^{-1} G_p\right) \det(G_k - \lambda I) \end{aligned}$$

Note that

$$\begin{aligned}
\delta(1) &= \det\left((F_p - 1) - F_k (G_k - \lambda I)^{-1} G_p\right) \det(G_k - I) \\
&= -f'(p) \det(G_k - I) \\
&= f'(p) \det(I - G_k)
\end{aligned}$$

We can show that $\det(I - G_k) > 0$, Thus

$$sign(\delta(1)) = sign\left(f'(p)\right)$$

QED.

B.1 An economy with perfectly sticky wages

In this section we describe a detailed economy where theorem 2 applies. To distinguish from Shimer, we assume that production technology is decreasing returns to scale. All other model setups are the same as the benchmark model in section 2. Specifically, the household-enterprenuer's problem is:

$$\max_{c,h,l,l^d,k} \sum_{t=1}^{\infty} U(c,h,l)$$

subject to

$$\begin{aligned}
c_t + p_t h_t + b_t + k_t &\leq w_t l_t + \pi_t + p_t h_{t-1} + q_t b_{t+1} + (1-\delta) k_{t-1} \\
\pi_t &= \max_{l_t^d} A \left(l_t^d\right)^{\gamma} k_{t-1}^{\alpha} - w_t l_t^d \\
q_t b_{t+1} + \theta w_t l_t^d &\leq \xi p_t h_t + \kappa k_t \\
0 &\leq l_t \leq l_0, c_t, h_t, k_t \geq 0, h_0, k_0 \text{ given}
\end{aligned}$$

Relative to the model described in section 2, we make two major changes. First, the households preference is given by:

$$U(c,h,l) = \frac{\left[\left[\omega c^{1-1/\eta} + (1-\omega) h^{1-1/\eta}\right]^{1/(1-1/\eta)}\right]^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}$$

.Second, the production technology is given by:

$$y = A \left(l_t^d\right)^{\gamma} k_{t-1}^{\alpha}$$

where we assume that

$$\gamma + \alpha \leq 1$$

.

Next, we wish to define an equilibrium with perfectly sticky wage. To do so, we first of all solve for the unconstrained steady state. That is, we drop the borrowing constraints for the firm and solve for the steady state.

$$\begin{aligned}
\beta\left(\alpha A l^{\gamma} k^{\alpha-1} + 1 - \delta\right) &= 1 \\
\omega c^{-1/\eta}\left[\omega c^{1-1/\eta} + (1-\omega) h_0^{1-1/\eta}\right]^{(1-\sigma)/(1-1/\eta)-1} &= w \chi l^{\frac{1}{\nu}} \\
c &= A l^{\gamma} k^{\alpha} - \delta k \\
w &= \gamma A l^{\gamma-1} k^{\alpha}
\end{aligned}$$

The first equation is the intertemporal first order condition for capital. The second one is the first order condition for labor. The third one is the resources constraints. The fourth one is the firm's hiring first order constraint. We have four equations and four unknowns (k, l, c, w). And we can solve for the unique unconstrained steady state denote it $(k_{ss}, l_{ss}, c_{ss}, w_{ss})$.

We will focus on competitive equilibrium where wage is exogenously fixed at $\bar{w} = w_{ss}$. We next state the definition of a competitive equilibrium:

Definition B.1.1 *A sticky-wage competitive equilibrium given w_{ss} is a sequence of allocations $\left\{c_t, k_{t+1}, h_{t+1}, l_t, l_t^d, b_t\right\}_{t=1}^{\infty}$ and sequence of prices $\{p_t, w_t, q_t\}_{t=1}^{\infty}$ such that:*

1. *Given prices, allocations solve the households problem*
2. *Housing and bond market clears every period $h = h_0, b = 0$*
3. *$w_t = w_{ss}$ for any t. Equilibrium labor is determined by the minimum of the labor demand and labor supply*

Given the definition for a competitive equilibrium, a sticky-wage steady state is a competitive equilibrium where capital stock k_t is time invariant.

We next state our main theorem, which implies the theorem 2:

Theorem 4 *Suppose $\eta < 1$. Then there exists an $\bar{\eta} < 1$ such that for any $\alpha + \gamma > \bar{\eta}$ and κ sufficiently small, there exists an interval U^{ξ} such that, if $\xi \in U^{\xi}$, then there exists more than 1 locally stable sticky-wage steady states given w_{ss}.*

Proof. The proof is similar to the proof of theorem 2. First of all, we can solve for the unconstrained steady state using the following equations, droping the borrowing constraint:

$$\begin{aligned}
\omega h_0^{-1/\eta} c^{1/\eta} + \beta p &= p \\
\beta\left(\alpha A l^{\gamma} k^{\alpha-1} + 1 - \delta\right) &= 1 \\
\omega c^{-1/\eta}\left[\omega c^{1-1/\eta} + (1-\omega) h_0^{1-1/\eta}\right]^{(1-\sigma)/(1-1/\eta)-1} &= w\chi l^{\frac{1}{\nu}} \\
c &= A l^{\gamma} k^{\alpha} - \delta k \\
w &= \gamma A l^{\gamma-1} k^{\alpha}
\end{aligned}$$

Suppose the system has a unique solution. Denote it $(k_{ss}, c_{ss}, p_{ss}, l_{ss}, w_{ss})$. Define

$$\xi_{ss} = \frac{w_{ss} l_{ss} - \kappa k_{ss}}{p_{ss} h_0}$$

■

For κ sufficiently small, $\xi_{ss} > 0$ and is thus feasible.

Next, we would like to show the following: for sufficiently small κ, given ξ_{ss}, there exists multiple sticky-wage steady states.

To show this, note that the unconstrained steady states is automatically a steady state because by definition of ξ_{ss}, it satisfies the borrowing constraint. Thus we only need to show that there exists another steady state and we are done. The system characterizing the steady state is given, as in the main text, by

$$
\begin{aligned}
\omega h_0^{-\frac{1}{\eta}} (c)^{\frac{1}{\eta}} + \beta p + \left(\frac{\gamma A k^{\alpha} (l)^{\gamma-1}}{w_{ss}} - 1 \right) \xi p &= p \\
\beta \left[A \alpha k^{\alpha-1} (l)^{\gamma} + (1-\delta) + \left(\frac{\gamma A k^{\alpha} (l)^{\gamma-1}}{w_{ss}} - 1 \right) \kappa \right] &= 1 \\
c + \delta k &= A k^{\alpha} (l)^{1-\alpha} \\
\min \left(\frac{\xi_{ss} p h_0 + \kappa k}{w_{ss}}, \left(\frac{\gamma A}{w_{ss}} \right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \right) &= l
\end{aligned}
$$

Note that we drops the households labor first order condition as the wage is sticky. We will verify later that at the other nontrivial steady state households first order condition satisfies

$$
\omega c^{-1/\eta} \left[\omega c^{1-1/\eta} + (1-\omega) h_0^{1-1/\eta} \right]^{(1-\sigma)/(1-1/\eta)-1} > w \chi l^{\frac{1}{\nu}}
$$

So the households would like to supply labor but are rationed out of the market.

We wish to show that apart from $(w_{ss}, l_{ss}, k_{ss}, c_{ss}, p_{ss})$, there exists another set of variables that solve these equations. First of all, we conjecture, and later verify that at the other steady state

$$
\frac{\xi_{ss} p h_0}{w} \leq \left(\frac{\gamma A}{w} \right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}}
$$

Thus we end up with a differentiable system of equations

$$
\begin{aligned}
\omega h_0^{-\frac{1}{\eta}} (c)^{\frac{1}{\eta}} + \beta p + \left(\frac{\gamma A k^{\alpha} (l)^{\gamma-1}}{w_{ss}} - 1 \right) \xi p &= p \\
\beta \left[A \alpha k^{\alpha-1} (l)^{\gamma} + (1-\delta) + \left(\frac{\gamma A k^{\alpha} (l)^{\gamma-1}}{w_{ss}} - 1 \right) \kappa \right] &= 1 \\
c + \delta k &= A k^{\alpha} (l)^{1-\alpha} \\
\frac{\xi_{ss} p h_0 + \kappa k}{w_{ss}} &= l
\end{aligned}
$$

Also since we take κ to be sufficiently small, we can further simplify the system considerably by looking at the limiting case where $\kappa = 0$(Rigorously speaking, we are interchanging limit and differentiation. To do so, we need to show that for a sequence of $\kappa_n \to 0$, the resulting sequence of functions $f'(p; \kappa_n)$ converges uniformly to $f'(p; 0)$. This is guaranteed by the fact that f is differntiable with respect to κ, and the derivative is bounded. See Walter Rudin's Principle of Mathematical Analysis, 3rd Edition, Theorem 7.17):

$$
\begin{aligned}
\omega h_0^{-\frac{1}{\eta}} (c)^{\frac{1}{\eta}} + \beta p + \left(\frac{\gamma A k^{\alpha} (l)^{\gamma-1}}{w_{ss}} - 1 \right) \xi p &= p \\
\beta \left[A \alpha k^{\alpha-1} (l)^{\gamma} + (1-\delta) \right] &= 1 \\
c + \delta k &= A k^{\alpha} (l)^{1-\alpha} \\
\frac{\xi_{ss} p h_0}{w_{ss}} &= l
\end{aligned}
$$

Plug $l = \frac{\xi_{ss} p h_0}{w_{ss}}$ back to the first three equations, we obtain

$$\begin{aligned}
\omega h_0^{-\frac{1}{\eta}} (c)^{\frac{1}{\eta}} + \beta p + \left(\frac{\gamma A k^{\alpha} \left(\frac{\xi_{ss} p h_0}{w_{ss}}\right)^{\gamma-1}}{w_{ss}} - 1 \right) \xi p &= p \\
\beta \left[A \alpha k^{\alpha-1} \left(\frac{\xi_{ss} p h_0}{w_{ss}} \right)^{\gamma} + (1-\delta) \right] &= 1 \\
c + \delta k &= A k^{\alpha} \left(\frac{\xi_{ss} p h_0}{w_{ss}} \right)^{1-\alpha}
\end{aligned}$$

Thus capital k can be expressed as a function of p from the second equation

$$\begin{aligned}
k &= \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{1}{\alpha-1}} \left(\frac{\xi p h_0}{\bar{w}} \right)^{\frac{-\gamma}{\alpha-1}} \\
&= N p^{\frac{\gamma}{1-\alpha}}
\end{aligned}$$

Where

$$N = \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{1}{\alpha-1}} \left(\frac{\xi h_0}{w_{ss}} \right)^{\frac{-\gamma}{\alpha-1}}$$

Thus by the third equation, consumption c is a function of p as well:

$$c = A \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\xi p h_0}{w_{ss}} \right)^{\frac{-\alpha\gamma}{\alpha-1}} \left(\frac{\xi p h_0}{w_{ss}} \right)^{\gamma} - \delta \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{1}{\alpha-1}} \left(\frac{\xi p h_0}{w_{ss}} \right)^{\frac{-\gamma}{\alpha-1}}$$

Thus

$$c = M p^{\frac{\gamma}{1-\alpha}}$$

Where

$$M = A \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\xi h_0}{\bar{w}} \right)^{\frac{-\gamma}{\alpha-1}} - \delta \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{1}{\alpha-1}} \left(\frac{\xi h_0}{\bar{w}} \right)^{\frac{-\gamma}{\alpha-1}}$$

Plug all of these into the first equation: Thus, p solves:

$$a \left(m p^{\frac{\gamma}{1-\alpha}} \right)^{\frac{1}{\eta}} + b p^{\frac{\gamma}{1-\alpha}} - c p = 0 \tag{B.15}$$

Where

$$a = \theta h_0^{-\eta}$$

$$\begin{aligned}
m &= A \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\xi_{ss} h_0}{w_{ss}} \right)^{\frac{-\gamma}{\alpha-1}} - \delta \left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{1}{\alpha-1}} \left(\frac{\xi_{ss} h_0}{w_{ss}} \right)^{\frac{-\gamma}{\alpha-1}} \\
b &= \xi_{ss} \frac{\gamma A}{w_{ss}^{\gamma}} \left(\left(\frac{\frac{1}{\beta} - 1 + \delta}{A\alpha} \right)^{\frac{1}{\alpha-1}} \left(\frac{\xi_{ss} h_0}{w_{ss}} \right)^{\frac{-\gamma}{\alpha-1}} \right)^{\alpha} (\xi_{ss} h_0)^{\gamma-1} \\
c &= 1 - \beta + \xi_{ss}
\end{aligned}$$

We wish to show that equality B.15 has multiple roots. As argued previously, one root is p_{ss} as the unconstrained steady state is automatically a steady state by the choice of ξ_{ss}. Also note that $p = 0$ is a trivial steady state. Next we would like to show the following two statements

1. $f'(p_{ss}) > 0$
2. $f'(0) > 0$

If the above two statements are true, then by continuity of f and the intermediate value theorem, we obtain another nontrivial steady state $p \in (0, p_{ss})$

To prove the first statement, note that

$$f'(p_{ss}) = \frac{1}{\eta}\frac{\gamma}{1-\alpha}a\left(mp_{ss}^{\frac{\gamma}{1-\alpha}}\right)^{\frac{1}{\eta}-1} mp_{ss}^{\frac{\gamma}{1-\alpha}-1} + \frac{\gamma}{1-\alpha}bp_{ss}^{\frac{\gamma}{1-\alpha}-1} - c$$

$$\begin{aligned} & mp_{ss}^{\frac{\gamma}{1-\alpha}} \\ = \quad & A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{\xi_{ss}h_0p_{ss}}{w_{ss}}\right)^{\frac{-\gamma}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{\xi_{ss}h_0p_{ss}}{w_{ss}}\right)^{\frac{-\gamma}{\alpha-1}}\end{aligned}$$

note that

$$\xi_{ss} = \frac{w_{ss}l_{ss}}{p_{ss}h_0}$$

$$\begin{aligned} mp_{ss}^{\frac{\gamma}{1-\alpha}} &= A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}(l_{ss})^{\frac{-\gamma}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}(l_{ss})^{\frac{-\gamma}{\alpha-1}} \\ &= \left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right](l_{ss})^{\frac{\gamma}{1-\alpha}} \\ &= Ak_{ss}^{\alpha}l_{ss}^{\gamma} - \delta k_{ss}\end{aligned}$$

$$\begin{aligned} bp_{ss}^{\frac{\gamma}{1-\alpha}-1} &= \xi_{ss}\frac{\gamma A}{w_{ss}^{\gamma}}\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{\xi_{ss}h_0}{w_{ss}}\right)^{\frac{-\gamma}{\alpha-1}}\right)^{\alpha}(\xi_{ss}h_0)^{\gamma-1}p_{ss}^{\frac{\gamma}{1-\alpha}-1} \\ &= \xi_{ss}^{1-\frac{\alpha\gamma}{\alpha-1}+\gamma-1}w_{ss}^{-\gamma+\frac{\alpha\gamma}{\alpha-1}}p_{ss}^{\frac{\gamma}{1-\alpha}-1}\gamma A\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}(h_0)^{\frac{-\gamma}{\alpha-1}}\right)^{\alpha}(h_0)^{\gamma-1} \\ &= \xi_{ss}^{\frac{\gamma}{1-\alpha}}w_{ss}^{-\frac{\gamma}{1-\alpha}}p_{ss}^{\frac{\gamma}{1-\alpha}}\gamma A\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}(h_0)^{\frac{-\gamma}{\alpha-1}}\right)^{\alpha}(h_0)^{\gamma-1}p_{ss}^{-1} \\ &= \left(\frac{l_{ss}}{h_0}\right)^{\frac{\gamma}{1-\alpha}}\gamma A\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}(h_0)^{\frac{-\gamma}{\alpha-1}}\right)^{\alpha}(h_0)^{\gamma-1}p_{ss}^{-1} \\ &= (l_{ss})^{\frac{\gamma}{1-\alpha}}\gamma A\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right)^{\alpha}p_{ss}^{-1}h_0^{-1} \\ &= \gamma Ak_{ss}^{\alpha}l_{ss}^{\gamma}p_{ss}^{-1}h_0^{-1}\end{aligned}$$

Thus

$$\begin{aligned}
f'(p_{ss}) &= \sigma\frac{\gamma}{1-\alpha}a\left(mp_{ss}^{\frac{\gamma}{1-\alpha}}\right)^{\sigma-1}mp_{ss}^{\frac{\gamma}{1-\alpha}-1}+\frac{\gamma}{1-\alpha}bp_{ss}^{\frac{\gamma}{1-\alpha}-1}-c \\
&= \sigma\frac{\gamma}{1-\alpha}a\left(\left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}-\delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right](l_0)^{\frac{\gamma}{1-\alpha}}\right)^{\sigma-1} \\
&\quad \left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}-\delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right](l_0)^{\frac{\gamma}{1-\alpha}}/p_{ss} \\
&\quad +\frac{\gamma}{1-\alpha}(l_0)^{\frac{\gamma}{1-\alpha}}\gamma A\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right)^{\alpha}h_0^{-1}p_{ss}^{-1}-1+\beta-\xi_{ss} \\
&= \frac{X}{p_{ss}}-\xi_{ss}-(1-\beta) \\
&= \frac{X}{p_{ss}}-\frac{w_{ss}l_0}{p_{ss}h_0}-(1-\beta) \\
&= \frac{X}{p_{ss}}-\frac{w_{ss}l_0}{p_{ss}h_0}-(1-\beta)
\end{aligned}$$

where

$$\begin{aligned}
X &= \frac{1}{\eta}\frac{\gamma}{1-\alpha}a\left(\left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}-\delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right](l_{ss})^{\frac{\gamma}{1-\alpha}}\right)^{\frac{1}{\eta}-1} \\
&\quad \left[A\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}}-\delta\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right](l_{ss})^{\frac{\gamma}{1-\alpha}} \\
&\quad +\frac{\gamma}{1-\alpha}(l_{ss})^{\frac{\gamma}{1-\alpha}}\gamma A\left(\left(\frac{\frac{1}{\beta}-1+\delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}\right)^{\alpha}h_0^{-1}
\end{aligned}$$

note that

$$k_{ss}=\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}l_{ss}^{-\frac{\gamma}{\alpha-1}}$$

Thus

$$k_{ss}^{\alpha}l_0^{\gamma}=\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha A}\right)^{\frac{\alpha}{\alpha-1}}l_{ss}^{-\frac{\gamma}{\alpha-1}}$$

Thus

$$\begin{aligned}
X &= \sigma\frac{\gamma}{1-\alpha}a\left(Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right)^{\sigma-1}\left[Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right] \\
&\quad +\frac{\gamma}{1-\alpha}\gamma Ak_{ss}^{\alpha}l_{ss}^{\gamma}h_0^{-1}
\end{aligned}$$

Thus

$$\begin{aligned}
f'\left(p_{ss}\right) &= \frac{\frac{1}{\eta}\frac{\gamma}{1-\alpha}a\left(Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right)^{\frac{1}{\eta}-1}\left[Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right]+\frac{\gamma}{1-\alpha}\gamma Ak_{ss}^{\alpha}l_{ss}^{\gamma}h_0^{-1}}{p_{ss}}-\frac{\gamma Ak_{ss}^{\alpha}l_{ss}h_0^{-1}}{p_{ss}}-(1-\beta)\\
&= \frac{\frac{1}{\eta}\frac{\gamma}{1-\alpha}a\left(Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right)^{\frac{1}{\eta}-1}\left[Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right]+\left(\frac{\gamma}{1-\alpha}-1\right)\gamma Ak_{ss}^{\alpha}l_{ss}^{\gamma}h_0^{-1}}{\frac{\theta h_0^{-\eta}\left(Ak_{ss}^{\gamma}l_{ss}^{\alpha}-\delta k_{ss}\right)^{\frac{1}{\eta}}}{1-\beta}}-(1-\beta)\\
&= (1-\beta)\left(\frac{\frac{1}{\eta}\frac{\gamma}{1-\alpha}a\left(Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right)^{\frac{1}{\eta}-1}\left[Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right]+\left(\frac{\gamma}{1-\alpha}-1\right)\gamma Ak_{ss}^{\alpha}l_{ss}^{\gamma}h_0^{-1}}{\theta h_0^{-\eta}\left(Ak_{ss}^{\gamma}l_{ss}^{\alpha}-\delta k_{ss}\right)^{\frac{1}{\eta}}}-1\right)\\
&= (1-\beta)\left(\frac{\theta h_0^{-\eta}\left(Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\delta k_{ss}\right)^{\frac{1}{\eta}-1}\left[\frac{1}{\eta}\frac{\gamma}{1-\alpha}Ak_{ss}^{\alpha}l_{ss}^{\gamma}-\sigma\frac{\gamma}{1-\alpha}\delta k_{ss}-Ak_{ss}^{\gamma}l_{ss}^{\alpha}+\delta k_{ss}\right]}{\theta h_0^{-\eta}\left(Ak_{ss}^{\gamma}l_0^{\alpha}-\delta k_{ss}\right)^{\frac{1}{\eta}}}\right)\\
&= (1-\beta)\left(\frac{\theta h_0^{-\eta}\left(c_{ss}\right)^{\frac{1}{\eta}}\left(\frac{1}{\eta}\frac{\gamma}{1-\alpha}-1\right)+\left(\frac{\gamma}{1-\alpha}-1\right)\gamma Ak_{ss}^{\alpha}l_{ss}^{\gamma}h_0^{-1}}{\theta h_0^{-\eta}\left(c_{ss}\right)^{\frac{1}{\eta}}}\right)
\end{aligned}$$

Thus, $f'\left(p_{ss}\right)>0$ if $\frac{1}{\eta}$ is sufficiently big. Note that for a given $\eta<1$, there exists $\frac{\gamma}{1-\alpha}$ sufficiently close to 1, or equivalently $\alpha+\gamma$ sufficiently close to 1, so that $f'\left(p_{ss}\right)>0$

To prove the second statement, lets evaluate

$$\begin{aligned}
& f'(0)\\
&= \lim_{p\to 0}\sigma\frac{\gamma}{1-\alpha}a\left(mp^{\frac{\gamma}{1-\alpha}}\right)^{\sigma-1}mp^{\frac{\gamma}{1-\alpha}-1}+\frac{\gamma}{1-\alpha}bp^{\frac{\gamma}{1-\alpha}-1}-c\\
&= \sigma\frac{\gamma}{1-\alpha}a\left(y_0\right)^{\sigma-1}mp^{\frac{\gamma}{1-\alpha}-1}+\frac{\gamma}{1-\alpha}bp^{\frac{\gamma}{1-\alpha}-1}-c
\end{aligned}$$

$$\begin{aligned}
& f'(0)\\
&= \frac{\gamma}{1-\alpha}bp^{\frac{\gamma}{1-\alpha}-1}-c
\end{aligned}$$

And we know that

$$b \text{ is positive}$$

So for p sufficiently small, this derivative is positive.

Therefore we have established that for $\xi=\xi_{ss}$ there exists multiple nontrivial steady states.

Next, we estabilish that for $\xi>\xi_{ss}$, but sufficiently close to ξ_{ss}, there exists multiple nontrivial steady states as well. This is obvious as $f(p)$ is continuous with respect to ξ. Therefore, for sufficiently small changes in ξ, we still have multiple nontrivial steady states.

To see this more precisely, we know that given $\xi=\xi_{ss}$, there exists $p_1<p_2$ such that $f\left(p_1;\xi=\xi_{ss}\right)>0$ and $f\left(p_2;\xi=\xi_{ss}\right)<0$. By continuity of f with respect to ξ, a small change in ξ still preverves that $f\left(p_1;\xi\right)>0$ and $f\left(p_2\right)<0$. This implies that multiple nontrivial steady states exist. Thus we can pick an interval U^{ξ} such that for any $\xi\in U^{\xi}$, multiple steady states exist.

The proof of local stability is very similar to the proof in theorem 2 and is hence delegated to the online appendix. QED.

The theorem highlights the difference between this paper and Shimer(2012). Shimer's result of multiple steady state only holds with constant returns to scale production technology. Here, I deliberately choose the production technology to be decreasing returns to scale and prove that there exists multiple steady states.

B.2 Computational Appendix

In this section I describe the functional equations characterzing the recursive competitive equilibrium and how I proceed to solve them. There are four functional equations that characterizes four equilibrium functions:

1. Law of motion for aggregate capital: $K' = \Phi(K, W_{-})$ where W_{-} is previous period wage.
2. Law of motion for wage: $W = W(K, W_{-})$
3. Land price function: $P = P(K, W_{-})$
4. Labor function: $L = L(K, W_{-})$

We have four functional equations, incorporating two occasionally binding constraints, to solve these functions. The first equation is the households first order condition for land.

The second equation is the firm's first order condition for investment:

$$\begin{aligned}
& U_c\left(AK^{\alpha}\left(L\left(K,W_{-}\right)\right)^{\gamma}\bar{h}_f^{1-\alpha-\gamma}+\left(1-\delta\right)K-\Phi\left(K,W_{-}\right)\right) \\
= \quad & \beta[\alpha A\left[\Phi\left(K,W_{-}\right)\right]^{\alpha-1}\left[L\left(\Phi\left(K,W_{-}\right),W\left(K,W_{-}\right)\right)\right]^{\gamma}\bar{h}_f^{1-\alpha-\gamma}+1-\delta \\
& +\frac{\kappa\gamma A\left[\Phi\left(K,W_{-}\right)\right]^{\alpha}\left[L\left(\Phi\left(K,W_{-}\right),W\left(K,W_{-}\right)\right)\right]^{\gamma-1}\bar{h}_f^{1-\alpha-\gamma}}{W\left(\Phi\left(K,W_{-}\right),W\left(K,W_{-}\right)\right)}-\kappa] \\
& U_c\left(A\left[\Phi\left(K,W_{-}\right)\right]^{\alpha}\left[L\left(\Phi\left(K,W_{-}\right),W\left(K,W_{-}\right)\right)\right]^{\gamma}+\left(1-\delta\right)\Phi\left(K,W_{-}\right)-\Phi\left(\Phi\left(K,W_{-}\right),W\left(K,W_{-}\right)\right)\right)
\end{aligned} \tag{B.16}$$

The third equation is households first order condition, taking into account the downward wage rigidity

$$W\left(K,W_{-}\right)=\max\left(\zeta W_{-},\frac{\chi\left(L\left(K,W_{-}\right)\right)^{\frac{1}{\nu}}}{U_c\left(AK^{\alpha}\left(L\left(K,W_{-}\right)\right)^{\gamma}\bar{h}_f^{1-\alpha-\gamma}+\left(1-\delta\right)K-\Phi\left(K,W_{-}\right)\right)}\right) \tag{B.17}$$

The fourth equation is firm's first order condition on hiring taking into account the borrowing constraint

$$L\left(K,W_{-}\right)=\min\left\{\left(\frac{\zeta W\left(K,W_{-}\right)}{\gamma A}\right)^{\frac{1}{\gamma-1}}K^{\frac{\alpha}{1-\gamma}},\frac{\zeta P\left(K,W_{-}\right)h_0+\kappa K}{W\left(K,W_{-}\right)}\right\} \tag{B.18}$$

Now, this is a system of four equations and we would like to solve four equilibrium functions from it. We employ a policy iteraction algorithm, adjusting for the occasionally binding constraint. Similar to Coleman(1990), and Bianchi(2013). To reduce the number of equations we need to solve, a crucial observation is that: given $\Phi(K, W_{-})$ and $P(K, W_{-})$, the labor market equilibrium can be solved seperately. Thus, we only need to run loops with respect to $\Phi(K, W_{-})$ and $P(K, W_{-})$ and solve $W(K, W_{-})$ and $L(K, W_{-})$ within each loop. Specifically:

1. Setup a state space $U^W \times U^K$. I consider $U^w = [0.8w_{ss}, w_{ss}]$, $U^K = [0.8k_{ss}, k_{ss}]$. This is sufficient for our purpose. I pick 80 grids for wage (evenly spaced) and 100 grids for capital, 50 of the grids are placed between $[0.9k_{ss}, 0.95k_{ss}]$ where there are strong nonlinearities. It is crucial to set up fine grids for the state variables, especially wage.
2. Make a good initial guess for $\Phi(K, W_{-})$ and $P(K, W_{-})$. Otherwise, the program would not converge properly, especially for low η.To do so, I first solve a frionless case and a case with perfectly wage stickiness and use them as the benchmark. Specifically, let $\Phi_f(K)$ and $P_f(K)$ denote the law of motion

and land price function a the frictionless case. Let $\Phi_s(K)$ and $P_s(K)$ denote the law of motion and land price function at perfect sticky wage case. I then conjecture:

$$\begin{aligned}\Phi^1(K,(1-\theta)w_{ss}) &= \theta\Phi_f(K)+(1-\theta)\Phi_s(K)\\ P^1(K,(1-\theta)w_{ss}) &= \theta P_f(K)+(1-\theta)P_s(K)\end{aligned}$$

Namely, I take a weighted average of the two benchmarks. The weights depend on how close the wage grid is to w_{ss}. Turns out this is a good initial guess that can get the program converge

3. Given $\Phi^n(K,W_-)$ and $P^n(K,W_-)$, we can solve for $W^n(K,W_-)$ and $L^n(K,W_-)$ from equation B.17 and B.18. Specifically, we solve for w,l given K,W_- and given $\Phi^n(K,W_-)$ and $P^n(K,W_-)$, from the following two equations:

$$\begin{aligned}w &= \max\left(\zeta W_-,\frac{\chi(l)^{\frac{1}{\nu}}}{U_c\left(AK^\alpha(l)^\gamma\bar{h}_f^{1-\alpha-\gamma}+(1-\delta)K-\Phi^n(K,W_-)\right)}\right)\\ l &= \min\left\{\left(\frac{\zeta w}{\gamma A}\right)^{\frac{1}{\gamma-1}}K^{\frac{\alpha}{1-\gamma}},\frac{\zeta P^n(K,W_-)h_0+\kappa K}{w}\right\}\end{aligned}$$

4. Given $\Phi^n(K,W_-)$, $P^n(K,W_-)$, $W^n(K,W_-)$ and $L^n(K,W_-)$ we can solve for $\Phi^{n+1}(K,W_-)$, $P^{n+1}(K,W_-)$ by iterating on equation **??** and B.16. To do so, we need to take the following steps:

   (a) Given any state variable (K,W_-), for any k',p, we can solve for the conditional wage function $w(k',p,K,W_-)$ and the conditional labor function $l(k',p,K,W_-)$ from equation B.17 and B.18:

$$\begin{aligned}w &= \max\left(\zeta W_-,\frac{\chi(l)^{\frac{1}{\nu}}}{U_c\left(AK^\alpha(l)^\gamma\bar{h}_f^{1-\alpha-\gamma}+(1-\delta)K-k'\right)}\right)\\ l &= \min\left\{\left(\frac{\zeta w}{\gamma A}\right)^{\frac{1}{\gamma-1}}K^{\frac{\alpha}{1-\gamma}},\frac{\zeta p h_0+\kappa K}{w}\right\}\end{aligned}$$

   (b) Next, solve for k',p, given that tomorrow the equilibrium functions are given by $\Phi^n(K,W_-)$, $P^n(K,W_-)$, $W^n(K,W_-)$ and $L^n(K,W_-)$ and today's labor market equilibrium is given by the conditional wage function $w(k',p,K,W_-)$ and the conditional labor function $l(k',p,K,W_-)$:

$$\begin{aligned}&U_c\left(AK^\alpha\left(l\left(k',p,K,W_-\right)\right)^\gamma\bar{h}_f^{1-\alpha-\gamma}+(1-\delta)K-k'\right)\\ =\;&\beta[\alpha A\left[k'\right]^{\alpha-1}\left[L^n\left(k',w\left(k',p,K,W_-\right)\right)\right]^\gamma\bar{h}_f^{1-\alpha-\gamma}+1-\delta\\ &+\frac{\kappa\gamma A\left[k'\right]^\alpha\left[L^n\left(k',w\left(k',p,K,W_-\right)\right)\right]^{\gamma-1}\bar{h}_f^{1-\alpha-\gamma}}{W^n\left(k',w\left(k',p,K,W_-\right)\right)}-\kappa]\\ &U_c\left(A\left[k'\right]^\alpha\left[L^n\left(k',w\left(k',p,K,W_-\right)\right)\right]^\gamma+(1-\delta)k'-\Phi^n\left(k',w\left(k',p,K,W_-\right)\right)\right)\end{aligned}$$

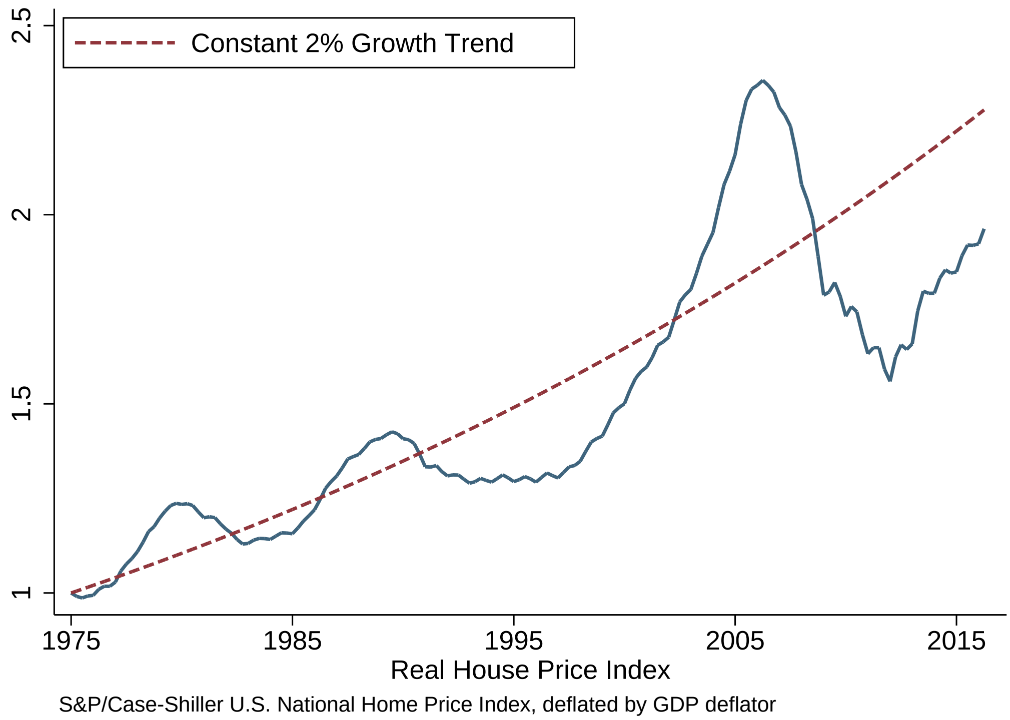
   The two equations solves p and k' given K,W_-. Thus we get $\Phi^{n+1}(K,W_-)$ and $P^{n+1}(K,W_-)$

5. Compare $[\Phi^n(K,W_-),P^n(K,W_-)]$ and $\left[\Phi^{n+1}(K,W_-),P^{n+1}(K,W_-)\right]$, if

$$\max\left|\left[\Phi^n(K,W_-),P^n(K,W_-)\right]-\left[\Phi^{n+1}(K,W_-),P^{n+1}(K,W_-)\right]\right|<10^{-8}$$

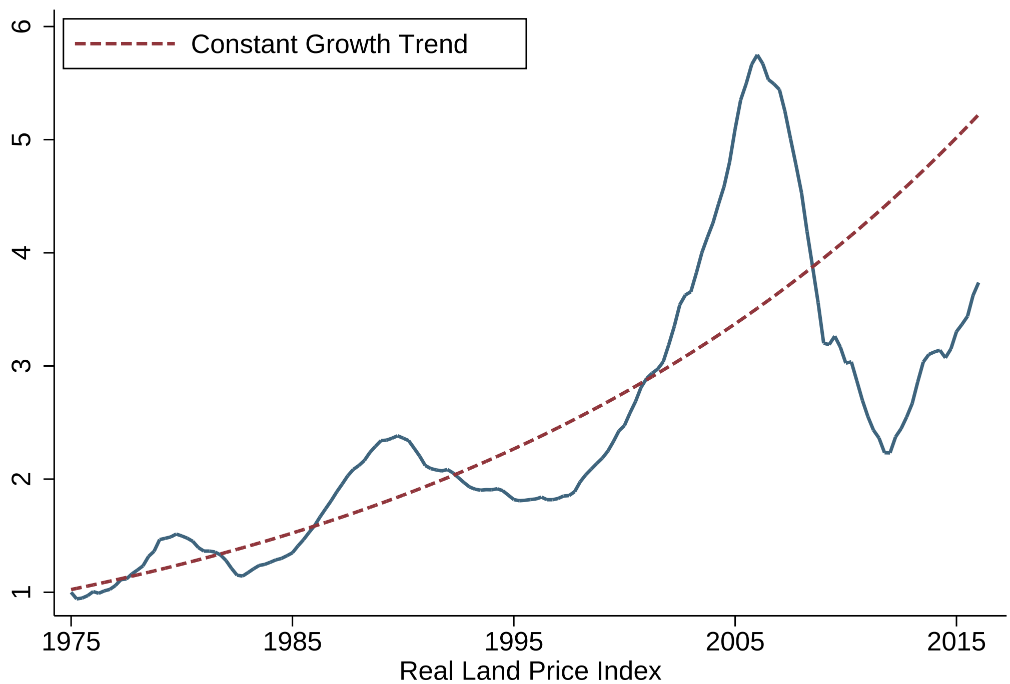
stop. Otherwise, set $n=n+1$, go back to step 4.

B.3 Various Graphs



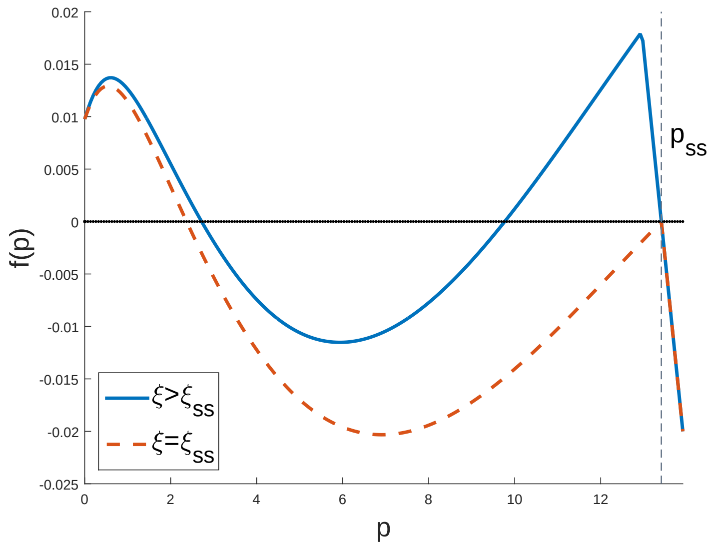
*This figure plots the S&P/Case-Shiller U.S. National Home Price Index, deflated by the GDP deflator, along with its constant growth trend. The growth rate is picked to be 2%, as in [**?**]. This is the average growth rate for real GDP per capita between 1947 and 2007. It is also close to the average growth rate for real house prices between 1975 and 2006 (see Figure 1 in Davis and Heathcote, 2007).*

Figure B.1: Real Housing Price and its trend



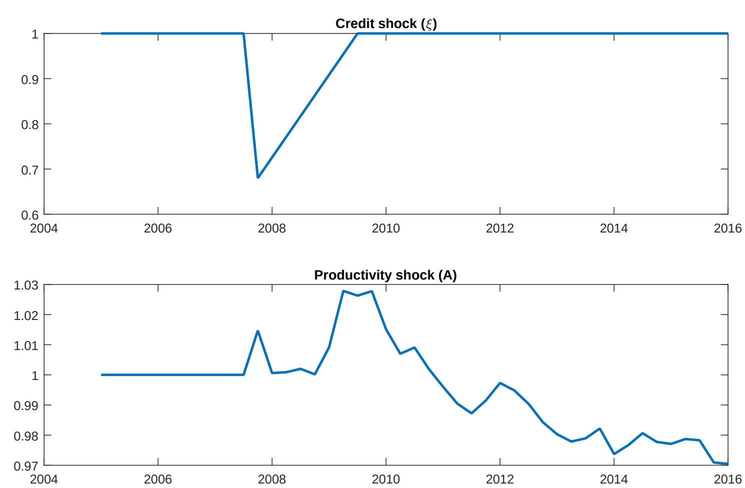
This figure plots the land price index available from the Lincoln Institute of Land Policy, which is constructed following [14], using Case-Shiller National Home Price Index and replacement cost of structures available from BEA. The data is available at http://datatoolkits.lincolninst.edu/subcenters/land-values/price-and-quantity.asp. For the constant growth trend, the growth rate of the trend is picked to match the average growth rate of real land price between 1975 and 1995.

Figure B.2: Real Land Price and its trend



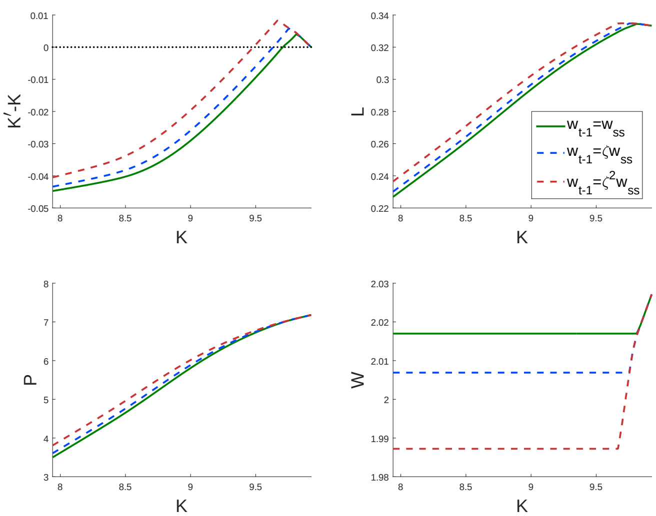
This figure illustrates the proof. I first consider the case where $\xi = \xi_{ss}$ (red dotted curve), where ξ_{ss} is such that at the unique frictionless steady state, firm's credit constraint holds with equality. In this case, I show that $f'(p_{ss}) > 0$ and $f(0) > 0$. Thus there exists multiple steady states. I then consider perturbation to $\xi > \xi_{ss}$ (blue curve) and show that results extends.

Figure B.3: Proof



This figure presents the timeseries of shocks I feed into the model

Figure B.4: Simulation



The figure plots various policy functions. The top left panel plots (normalized) law of motion for capital. Note that it is S-shaped conditional on previous period wage. The top left panel plots equilibrium labor function. The bottom left plots equilibrium land price function. The bottom right plots equilibrium wage function. plots of different colors are conditional on different level of previous-period wages.

Figure B.5: Policy Functions